Summary: Chapter 11

1. THE BIG PICTURE: Understand that functions (see note below) can be expressed as a polynomial:

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} a_n(x-a)^n$$

(Note: Not all functions can be expressed this way, we'll discuss the technical details in class)

2. Towards this goal, we broke this big problem into several concepts which evolved in the following way:

$$\text{SEQUENCES} \to \text{SERIES} \to \text{POWER SERIES} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty} \to \sum_{n=1}^{\infty} a_n \to \sum_{n=1}^{\infty} \text{Expression in } n, x$$

With the main questions evolving:

What does it mean to take the limit of a_n? $\to$ What does it mean to take an infinite sum? $\to$ For what values of x will we get a meaningful infinite sum?

3. To guide our intuition, we set up some guideposts- some templates by which we will try to understand more complex examples:

- Template Sequence: $a_n = r^n$ Converges to 0 if $|r| < 1$ (Otherwise, divergent)
- Template Series:
  - The Geometric Series: $\displaystyle\sum_{n=k}^{\infty} ar^n$ If $|r| < 1$, converges to
  $$\frac{\text{First Term}}{1 - \text{Ratio}} = \frac{ar^k}{1-r}$$
  - The $P-$series: $\displaystyle\sum \frac{1}{n^p}$ converges if $p > 1$
  (Compare with Ch. 7.8, $f(x) = (1/x)^p$).
    * The Harmonic Series: $\sum \frac{1}{n}$ diverges.
    * The Alternating Harmonic Series $\sum (-1)^n \frac{1}{n}$ converges conditionally.

  Unlike the Geometric Series, we don't have a nice formula by which we can compute the actual sum to which these series will converge. We can estimate the sum (won't be tested), or we can use an associated Power Series.
- Template Power Series:
  - Exponential: $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
  - Geo-style: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (if $|x| < 1$)

4. Answering the big issues:

   (a) Take the limit of a_n: We saw that we can connect the sequence a_n to the function $f(n)$. All the methods we had in Chapters 2 and 4 can now be applied to find a "horizontal asymptote" for the sequence. We added the template sequence r^n.

   (b) What does it mean to take an infinite sum?

      i. If $s_n = \sum_{i=1}^{n} a_i$, then we define $\sum_{i=1}^{\infty} a_i$ as the limit of s_n. If the limit exists, we say that $\sum_{i=1}^{\infty} a_i$ converges. Otherwise, the sum diverges.

      ii. Connection to Sequences: (The Divergence Test) If we are considering $\sum a_n$, and a_n does not go to zero, the sum diverges. On the other hand, if $a_n \to 0$, the sum may or may not converge. This is why we now bring up the convergence tests.
      CONVERGENCE TESTS answer the question: How fast do the terms of the series go to zero? This takes the bulk of Chapter 11, so we'll address this in detail in the next section.

(c) Conditional v. Absolute Convergence:

Absolute convergence is a beautiful thing: It says that the infinite series behaves like a finite sum- rearrangements of the terms do not effect the final outcome.

Conditional convergence is a monstrosity! Through an appropriate rearrangement of the terms of the sum, the sum will converge to ANY real number!!

If a series is absolutely convergent, then it is convergent.

(d) Find x so that the power series converges. This is an abstraction of a regular series, in that we introduce a new variable x. The idea is that, whenever we replace x by an actual number, the result is a regular series. Thus, to say that a power series converges means that, by substituting the proper values of x, the corresponding series converges. So, all our convergence tests will apply to these problems.

(e) The final question: Find the coefficients so that $f(x)$ can be expressed as a power series (the Taylor series). At a base point, $x = a$, we can write $f(x)$ as:

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Implicit in this formula is that we can take all derivatives of f. An interesting question is whether or not we can always do this (even when the derivatives all exist). It turns out that we could have problems- although the functions we consider will all be OK. Since we have this problem, we can use a new vocabulary term: A function is said to be *analytic* if it can be represented by it's Taylor series.

Note the special cases:

- $f(x) \approx f(a)$ (a constant)
- $f(x) \approx f(a) + f'(a)(x-a)$ The linearization of f at $x = a$
- $f(x) \approx f(a) + f'(a)(x-1) + \frac{f''(a)}{2}(x-a)^2$ A quadratic approximation of f at $x = a$ (Fits a parabola to f)

Understanding the Convergence Tests:

First and foremost, check that the terms $a_n \to 0$. If not, we're done (the series diverges). If so, the series may or may not converge. Note that we will only be interested in the convergence question- actually computing the sum to which the series will converge is a more complicated question (with the exception of the Geometric Series or a Power Series).

If the terms of the series go to zero, we check for absolute convergence first- which means that we'll treat our series as a positive series:

TESTS FOR POSITIVE SERIES:

The main issue here is HOW FAST DO OUR TERMS GO TO ZERO?

1. Direct comparison of unknown series $\sum b_n$ with a known series $\sum a_n$.

   (a) If $b_n \leq a_n$ for all n and $\sum a_n$ converges, so does b_n (The terms in $\sum b_n$ go to zero faster than our known series).

   If $b_n \geq a_n$ for all n and $\sum a_n$ diverges, so does b_n (The terms in $\sum b_n$ go to zero slower than our known series).

   (b) COMMENTS: This method requires a delicate touch! It's probably better to use the Limit Comparison, next.

2. The Limit Comparison of unknown series $\sum b_n$ with a known series $\sum a_n$. This uses the following idea- I'd like for you to understand what's happening, rather than memorizing all of the cases!

   - If $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n} = c$, where $c > 0$ and c is finite, we say that a_n and b_n go to zero *at the same rate*. Thus, the corresponding series will converge or diverge together.
   - If $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n} = 0$, this means that b_n is going to zero faster than a_n. If the series for a_n converges, so will the series for b_n. If the series for a_n diverges, we can say nothing about $\sum b_n$.
   - If $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$, this means that a_n is going to zero faster than b_n. If the series for a_n converges, we can say nothing about $\sum b_n$. If the series for a_n diverges, so does the series for b_n.
   - COMMENT: This method is much more robust to approximation than the direct comparison- It is a favorite!! The main issue in using this is that we need a template series to do the comparisons.

3. The Integral Test: Here is where we formally connect the concept of an infinite sum to an improper integral (Ch 7.8). That is, the following two quantities converge or diverge together (if f is positive and decreasing):

$$\int_1^\infty f(x)\,dx \quad \text{and} \quad \sum_{n=1}^{\infty} a_n, \text{ with } a_n = f(n)$$

   COMMENT: Nice idea, and very useful if a_n looks like a function that we can integrate (but this is usually not the case). In the previous methods (Direct and Limit Comparison) we needed a known series to do the comparing, but not in this case.

4. Self Comparison 1: The Ratio Test

   If $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, we say that the series $\sum a_n$ converges like a geometric series with ratio L.

   If $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$, we say that the series $\sum a_n$ diverges.

   If $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the test fails (we cannot conclude anything).

5. Self Comparison 2: The Root Test

   If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L < 1$, we say that the series $\sum a_n$ converges like a geometric series with ratio L.

   If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L > 1$, we say that the series $\sum a_n$ diverges.

   If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$, the test fails (we cannot conclude anything).

   COMMENTS ON RATIO and ROOT TESTS:

   Using the Ratio and Root Tests on a series whose terms go to zero like the terms of a $p-$series will both fail. The root test is really only used when you have a complicated expression being raised to an n^{th} power. The Ratio Test works well with things involving $n!$.

Alternating Series Test:

If the absolute value of the terms are *decreasing* to zero, and the terms are *alternating* in sign, then the infinite sum will converge.

Other Methods and Algebra:

1. To find a limit of a sequence, we have the following choices:
   - Put it into the form ar^n. It goes to zero if $|r| < 1$.
   - Divide numerator and denominator by n raised to a power.
   - Rationalize: Given something like "$\sqrt{\quad}$ − something", multiply by the fraction $\frac{\sqrt{\quad}+\text{something}}{\sqrt{\quad}+\text{something}}$
   - L'Hospital's Rule (A favorite!). You must have a form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$
     - $f(x)g(x)$ can be written as $\frac{f(x)}{1/g(x)}$
     - $f(x)^{g(x)}$ can be written as $\mathrm{e}^{g(x)\ln(f(x))}$