Review: Exam 2

1. Goals for this portion of the course:

   - Be able to compute the area between curves, the volume of solids of revolution, and understand the mean value of a function. We had three basic volumes: Disk, Washer and Shell.
   - Stimulate mental connections between geometry and calculus- Given a picture, write down the formula for the given area or volume of a solid. Apply algebra if needed to obtain the required information (i.e., points of intersection). Also, be able to go from the integral of an area/volume to a geometric picture.
   - Be able to finish the integration to obtain a final solution.
   - Understand and apply u, du substitution to find antiderivatives (and evaluate the integral).

2. Material to be covered: 6.1, 6.2, 6.3, 6.5 plus review of u, du substitution.

Review Questions:

1. Find the area of the region bounded between the curves:

   (a) $y = \mathrm{e}^x - 1$, $y = x^2 - x$, $x = 1$

   Top function: $\mathrm{e}^x - 1$, bottom function: $x^2 - x$, so the area is $\int_0^1 \mathrm{e}^x - 1 - (x^2 - x)\,dx$ which is $\mathrm{e} - (11/6)$

   (b) $x - 2y + 7 = 0$, $y^2 - 6y - x = 0$

   The rightmost function is the line, leftmost is the parabola, and integrate in y:

   $$\int_1^7 (2y - 7) - (y^2 - 6y)\,dy = 36$$

2. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis. You may use any method.

   (a) $y = \sqrt{x-1}$, $y = 0$, $x = 3$, about the x-axis.

   Use disks with radius $\sqrt{x-1}$:

   $$\int_1^3 \pi(\sqrt{x-1})^2\,dx = 2\pi$$

   (b) $y = \mathrm{e}^{-2x}$, $y = 1 + x$, $x = 1$, about the x-axis.

   One point of intersection between $y = \mathrm{e}^{-2x}$ and $y = 1 + x$ is at $(0, 1)$ since that is the $y-$intercept of the line, and we know that $\mathrm{e}^0 = 1$. The lines $y = 1 + x$ and $x = 1$ intersect at $(1, 2)$, and e^{-2x} intersects $x = 1$ at $(1, \mathrm{e}^{-2})$. The area is the small region between these points of intersection.

Using washers, the inside radius is e^{-2x} and the outside radius is $1+x$. Therefore, the volume is:

$$\int_0^1 \pi\left[(1+x)^2 - (e^{-2x})^2\right]\,dx = \pi\int_0^1 1+2x+x^2-e^{-4x}\,dx =$$

$$\left(x+x^2+\frac{x^3}{3}+\frac{1}{4}e^{-4x}\right|_0^1 = \pi\left(\frac{25}{12}+\frac{1}{4e^4}\right)$$

(c) $y=x^3$, $y=8$, $x=0$, about the $y-$axis.
Use disks and integrate with respect to y:

$$\int_0^8 (y^{1/3})^2\,dy = \frac{96\pi}{5}$$

3. If f is continuous, and $\int_1^3 f(x)\,dx = 8$, so that f takes on the value 4 at least once on the interval $[1,3]$.

***TYPO: "so that" should be "show that"

From the Mean Value Theorem for integrals, there is a c between a and b so that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3-1}\cdot 8 = 4$$

4. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis. You may use any method.

(a) $y=x^3$, $y=x^2$, about $y=1$

$$\text{Washers: } \int_0^1 \pi\left[(1-x^3)^2-(1-x^2)^2\right]\,dx$$

(b) $y=x^3$, $y=8$, $x=0$ about $x=2$.

$$\text{Shells: } \int_0^2 2\pi(8-x^3)(2-x)\,dx$$

5. Each integral represents the volume of a solid. Describe the solid (with words and/or pictures).

(a) $\int_0^{\pi/2} 2\pi x\cos(x)\,dx$
Rotate the region under $\cos(x)$ (and above the x-axis) for $0 \le x \le \pi/2$ about the $y-axis$

(b) $\int_0^{\pi/2} 2\pi \cos^2(x)\, dx$

Rotate the region under $\sqrt{2}\cos(x)$ (above the x-axis) with $0 \le x \le \pi/2$.

(c) $\int_0^1 \pi\left[(2-x^2)^2 - (2-\sqrt{x})^2\right]\, dy$

***TYPO: The dy should be dx

The solid is found by rotating the region between $y = x^2$ and $y = \sqrt{x}$ $(0 \le x \le 1)$ about the line $x = 2$.

6. Find the average value of $f(x) = x^2\sqrt{1+x^2}$ on the interval $[0, 2]$.

***TYPO: $\sqrt{1+x^2}$ should be $\sqrt{1+x^3}$

$$f_{\text{avg}} = \frac{1}{2}\int_0^2 x^2\sqrt{1+x^3}\, dx$$

Use substitution with $u = 1 + x^3$, $du = 3x^2\, dx$

$$f_{\text{avg}} = \frac{1}{6}\int_1^9 \sqrt{u}\, du = \frac{26}{9}$$

7. Let R be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$. Calculate the following quantities:

(a) The area of R.

$$\int_0^1 2x - x^2 - x^3\, dx = \frac{5}{12}$$

(b) The volume obtained by revolving R about the $x-$axis.

A cross section is a washer with inner radius x^3 and outer radius $2x - x^2$

$$\int_0^1 \pi\left[(2x-x^2)^2 - x^6\right]\, dx = \frac{41\pi}{105}$$

(c) The volume obtained by revolving R about the $y-$axis.

Using shells, the radius is x and the height is $2x - x^2 - x^3$, so we get:

$$\int_0^1 2\pi x(2x - x^2 - x^3)\, dx = \frac{13\pi}{30}$$

8. Find the numbers b so that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

The requirement is that

$$\frac{1}{b-0}\int_0^b 2 + 6x - 3x^2\, dx = 3$$

Write out the left hand side of that equation to get:

$$2 + 3b - b^2 = 3 \Rightarrow b = \frac{3 \pm \sqrt{5}}{2}$$

both of which are valid, since both are positive.

9. Let R_1 be the region bounded by $y = x^2$, $y = 0$, and $x = b$, with $b > 0$. Let R_2 be the region bounded by $y = x^2$, $x = 0$, and $y = b^2$ (Same b).

   (a) Is there a b so that R_1 and R_2 have the same area?

   The area of R_1 is $\int_0^b x^2\,dx = \frac{1}{3}b^3$. The area of R_2 is $\int_0^{b^2} \sqrt{y}\,dy = \frac{2}{3}b^3$. Therefore, no value of $b > 0$ will make these equal.

   (b) Is there a value of b so that R_1 sweeps out the same volume if R_1 is rotated about the $x-$axis versus rotating R_1 about the $y-$axis?

   The first volume is found with disks: $\int_0^b \pi(x^2)^2\,dx = \frac{1}{5}\pi b^5$. The second volume can be found with shells:

$$\int_0^b 2\pi x(x^2)\,dx = \frac{1}{2}\pi b^4$$

   So setting the volumes equal:

$$\frac{1}{5}\pi b^5 = \frac{1}{2}\pi b^4 \Rightarrow 2b = 5 \Rightarrow b = \frac{5}{2}$$

   (c) Is there a value of b so that R_1 and R_2 sweep out the same volume when rotated about the $y-$axis?

   The volume of R_1 about the $y-$axis was computed already as $\frac{1}{2}\pi b^4$. The volume of R_2 about the $y-$axis is given by (disks):

$$\int_0^{b^2} \pi(\sqrt{y})^2\,dy = \frac{1}{2}\pi b^4$$

   Therefore, the two volumes are always equal (for all values of b)!

10. Evaluate:

   (a) $\int x\sqrt{x-1}\,dx$ Let $u = x - 1$, so $du = dx$. Upon substitution, we see an extra x, so we can substitute $u + 1 = x$, and obtain:

$$\int (u+1)\sqrt{u}\,du = \int u^{3/2} + u^{1/2}\,du = \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C =$$

$$\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$

(b) $\int \frac{x+1}{x^2+2x}\,dx$ Let $u = x^2+2x$, and get:

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+2x| + C$$

(c) $\int \frac{x}{\sqrt[4]{x+2}}\,dx$ Let $u = x+2$ (Note that $x = u-2$):

$$\int \frac{u-2}{\sqrt[4]{u}}\,du = \int u^{3/4} - 2u^{-1/4}\,du = \frac{4}{7}u^{7/4} - \frac{8}{3}u^{3/4} + C =$$

$$\frac{4}{7}(x+2)^{7/4} - \frac{8}{3}(x+2)^{3/4} + C$$

(d) $\int \frac{x^2}{\sqrt{1-x}}\,dx$ Let $u = 1-x$. (Note that $x = 1-u$ so that $x^2 = (1-u)^2$), and:

$$-\int \frac{(1-u)^2}{\sqrt{u}}\,du = -\int u^{-1/2} - 2u^{1/2} + u^{3/2}\,du =$$

$$-2u^{1/2}+\frac{4}{3}u^{3/2}+\frac{2}{5}u^{5/2}+C = -2(1-x)^{1/2}+\frac{4}{3}(1-x)^{3/2}+\frac{2}{5}(1-x)^{5/2}+C$$