Integration Practice Solutions

1. $\int \frac{t^2\cos(t^3-2)}{\sin^2(t^3-2)}\,dt$

First, let $u = t^3-2$, $du = 3t^2\,dt$. Then we have:

$$\int \frac{t^2\cos(t^3-2)}{\sin^2(t^3-2)}\,dt = \frac{1}{3}\int \frac{\cos(u)}{\sin^2(u)}\,du$$

Now let $w=\sin(u)$, so $dw=\cos(u)\,du$, and we get

$$\frac{1}{3}\int \frac{\cos(u)}{\sin^2(u)}\,du = \frac{1}{3}\int \frac{1}{w^2}\,dw = -\frac{1}{3}w^{-1} =$$

$$-\frac{1}{3}\frac{1}{\sin(t^2-3)}+C$$

2. $\int \frac{3}{3x-1}\,dx$

Let $u=3x-1$, $du=3\,dx$, and

$$\int \frac{3}{3x-1}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|3x-1|+C$$

3. $\int \frac{x}{x^2+4}\,dx$

Use $u=x^2+4$, $du=2x\,dx$ to get:

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+4|+C$$

4. $\int \frac{1}{4-3(x-2)^2}\,dx$

From the list of identities, we can try the substitution suggested for something of the form a^2-x^2, so we'll try to use:

$$\sqrt{3}(x-2)=2\sin(\theta)$$

so that

$$4-3(x-2)^2 = 4(1-\sin^2(\theta)) = 4\cos^2(\theta),\ \sqrt{3}\,dx = 2\cos(\theta)\,d\theta$$

5. $\int \frac{6}{x^2+5}\,dx$

Use the sheet of integrals, $\int \frac{1}{x^2+a^2}\,dx$:

$$\int \frac{6}{x^2+5}\,dx = \frac{6}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$$

6. $\int \frac{x-1}{x^2+3}\,dx$ First rewrite the integral:

$$\int \frac{x-1}{x^2+3}\,dx = \int\frac{x}{x^2+3}\,dx + \int\frac{1}{x^2+3}\,dx$$

For the first integral, use a u, du substitution with $u = x^2+3$, $du = 2x\,dx$. For the second integral, use the table of integrals:

$$\int \frac{x}{x^2+3}\,dx+\int\frac{1}{x^2+3}\,dx = \frac{1}{2}\int\frac{1}{u}\,du - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) =$$

$$\frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$$

7. $\int \frac{x-1}{x^2-3}\,dx$

Note the difference between this and the previous problem. Now we have to factor and do partial fractions.

$$\frac{x-1}{x^2-3} = \frac{A}{x-\sqrt{3}} + \frac{B}{x+\sqrt{3}} \Rightarrow A = \frac{\sqrt{3}-1}{2\sqrt{3}},\ B = \frac{\sqrt{3}+1}{2\sqrt{3}}$$

Although this is a little messy, the integrals are straightforward:

$$\int\frac{x-1}{x^2-3}\,dx = \frac{\sqrt{3}-1}{2\sqrt{3}}\int\frac{1}{x-\sqrt{3}}\,dx + \frac{\sqrt{3}+1}{2\sqrt{3}}\int\frac{1}{x+\sqrt{3}}\,dx$$

To obtain:

$$\int\frac{x-1}{x^2-3}\,dx = \frac{\sqrt{3}-1}{2\sqrt{3}}\ln|x-\sqrt{3}| + \frac{\sqrt{3}+1}{2\sqrt{3}}\ln|x+\sqrt{3}|+C$$

8. $\int x e^{2x}\,dx$ Use integration by parts in a table to get:

$$\frac{1}{2}e^{2x} - \frac{1}{4}e^{2x}+C$$

9. $\int \sin^3(2t)\,dt$ Because we have an odd power of sine, we should try to set up a u, du substitution:

$$\int \sin^2(2t)\ \sin(2t)\,dt = \int (1-\cos^2(2t))\ \sin(2t)\,dt$$

Let $u=\cos(2t)$, $du = -2\sin(2t)\,dt$. Then

$$\int (1-\cos^2(2t))\ \sin(2t)\,dt = -\frac{1}{2}\int(1-u^2)\,du =$$

$$-\frac{1}{2}(u-\frac{1}{3}u^3)+C = -\frac{1}{2}\cos(2t)+\frac{1}{6}\cos^3(2t)+C$$

10. $\int x\sin(4x)\,dx$ Use integration by parts with a table:

$$-\frac{1}{4}\cos(4x)+\frac{1}{16}\sin(4x)+C$$

11. $\int \sin^{-1}(x)\,dx$ Use integration by parts with $u = \sin^{-1}(x)$, $du = \frac{1}{\sqrt{1-x^2}}$ and $dv = dx$, $v = x$ so that:

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int\frac{x}{\sqrt{1-x^2}}\,dx =$$

$$x\sin^{-1}(x)+\sqrt{1-x^2}+C$$

12. $\int \frac{Ax+B}{x^2+1}\,dx$ (Assume A, B are constants)

$$A\int\frac{x}{x^2+1}\,dx + B\int\frac{1}{x^2+1}\,dx = \frac{A}{2}\ln|x^2+1| + B\tan^{-1}(x)$$

13. $\int \frac{3x-2}{(x^2+2)^2}\,dx$

$$3\int\frac{x}{(x^2+2)^2}\,dx - 2\int\frac{1}{x^2+2)^2}\,dx$$

For the first integral, let $u = x^2+2, du = 2x\,dx$. For the second integral, let's try $x=\sqrt{2}\tan(\theta)$, so $dx = \sqrt{2}\sec^2(\theta)$. Putting in these substitutions, we get:

$$\frac{3}{2}\int u^{-2}\,du - 2\int\frac{\sqrt{2}\sec^2(\theta)}{4\sec^4(\theta)}\,d\theta = -\frac{3}{2}u^{-1} - \frac{\sqrt{2}}{2}\int\frac{1}{\sec^2(\theta)}\,d\theta$$

$$-\frac{3}{2}\frac{1}{x^2+2} - \frac{\sqrt{2}}{2}\int\cos^2(\theta)\,d\theta$$

I leave off a couple of details, but we should get to:

$$-\frac{3}{2}\frac{1}{x^2+2} - \frac{\sqrt{2}}{4}\theta + \frac{\sqrt{2}}{8}\sin(2\theta)$$

We convert back to x by using a triangle and the identity:

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$-\frac{3}{2(x^2+2)} - \frac{\sqrt{2}}{4}\tan^{-1}(x/\sqrt{2}) - \frac{\sqrt{2}}{4}\frac{x}{\sqrt{x^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2+2}}$$

14. $\int \frac{2}{\sqrt{9-5x^2}}\,dx$ Two ways to do this- directly or by using the formula on the back of our sheet.

To use the formula, notice that, if $u=\sqrt{5}x$, then we can re-write:

$$\int \frac{2}{\sqrt{9-5x^2}}\,dx = \frac{2}{\sqrt{5}}\int \frac{1}{\sqrt{9-u^2}}\,du$$

Use the table:

$$\frac{2}{\sqrt{5}}\int \frac{1}{\sqrt{9-u^2}}\,du = \frac{2}{\sqrt{5}}\sin^{-1}(u/3) = \frac{2}{\sqrt{5}}\sin^{-1}(\sqrt{5}x/3)+C$$

Doing it directly, let $\sqrt{5}\,x = 3\sin(\theta)$, and we get:

$$\frac{2}{\sqrt{5}}\int d\theta = \frac{2}{\sqrt{5}}\theta + C = \frac{2}{\sqrt{5}}\sin^{-1}(\sqrt{5}x/3)+C$$

15. $\int \frac{3}{3x^2+1}\,dx$

To get something directly from the table, let $u=\sqrt{3}\,x$. Then

$$\int \frac{3}{3x^2+1}\,dx = \sqrt{3}\int \frac{1}{u^2+1} = \sqrt{3}\tan^{-1}(u) = \sqrt{3}\tan^{-1}(\sqrt{3}\,x)+C$$

16. $\int \frac{1}{x^3+x^2-2x}\,dx$ First factor the denominator, then use partial fractions:

$$\frac{1}{x^3+x^2-2x} = \frac{1}{x(x+2)(x-1)} = \frac{(-1/2)}{x}+\frac{(1/6)}{x+2}+\frac{(1/3)}{x-1}$$

Then:

$$\int \frac{1}{x^3+x^2-2x}\,dx = -\frac{1}{2}\ln|x|+\frac{1}{6}\ln|x+2|+\frac{1}{3}\ln|x-1|+C$$

17. $\int \frac{x^2+3}{x^3+2x}\,dx$ Again, factor the denominator and use partial fractions:

$$\int \frac{x^2+3}{x^3+2x}\,dx = \frac{3}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{x}{x^2+2}\,dx$$

For the second integral, use a $u=x^2+2, du=2x\,dx$ substitution to get:

$$\frac{3}{2}\ln|x| - \frac{1}{4}\ln|x^2+4|+C$$

18. $\int x^3\sqrt{x^2+4}\,dx$ Use the following computations:

$$\begin{array}{lll} x=2\tan(\theta) & x^2+4=4\tan^2(\theta)+4 & x^3=2^3\tan^3(\theta) \\ dx=2\sec^2(\theta) & \sqrt{x^2+4}=2\sec(\theta) & \end{array}$$

and we get:

$$2^5\int \tan^3(\theta)\sec^3(\theta)\,d\theta = 2^5\int \frac{\sin^3(\theta)}{\cos^6(\theta)}\,d\theta = 2^5\int \frac{\sin^2(\theta)}{\cos^6(\theta)}\sin(\theta)\,d\theta =$$

$$2^5\int \frac{1-u^2}{u^6}\,du = 2^5\left(-\frac{1}{5}u^{-5}+\frac{1}{3}u^{-3}\right)+C = \frac{-2^5}{5}\sec^5(\theta)+\frac{2^5}{3}\sec^3(\theta)+C =$$

From a triangle, get that $\sec(\theta)=\frac{\sqrt{x^2+4}}{2}$, so that substituting it back in:

$$\frac{-2^5}{5}\left(\frac{\sqrt{x^2+4}}{2}\right)^5 + \frac{2^5}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 + C$$

Which we could simplify a bit more, but this is fine.

19. $\int \sqrt{2x-x^2}\,dx$ This one we did in class. First complete the square:

$$2x-x^2 = 1-(x-1)^2$$

and use: $x-1=\sin(\theta)$, $dx=\cos(\theta)\,d\theta$. Now we obtain $\int \cos^2(\theta)\,d\theta$ and use the double angle formula to get

$$\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)$$

which is

$$\frac{1}{2}\theta+\frac{1}{2}\sin(\theta)\cos(\theta) = \frac{1}{2}\sin^{-1}(x-1)+\frac{1}{2}(x-1)\sqrt{2x-x^2}+C$$

20. $\int \frac{u^3+1}{u^3-u^2}\,du$ Do long division and partial fractions to rewrite:

$$\frac{u^3+1}{u^3-u^2} = 1-\frac{1}{u}-\frac{1}{u^2}+\frac{2}{u-1}$$

so that

$$\int \frac{u^3+1}{u^3-u^2}\,du = u-\ln|u|+\frac{1}{u}+2\ln|u-1|+C$$