Take Home Quiz Solutions

1. Assigned April 3:

(a) Begin with a triangle, with an angle θ whose length of the side opposite is x, the hypotenuse is 1, and the length of the adjacent side is $\sqrt{1-x^2}$. Then we have:

$$\sin(\theta) = x \quad \cos(\theta) = \sqrt{1-x^2}, \quad \cos(\theta)\,d\theta = dx$$

and if $x = 0, \theta = 0$, if $x = \frac{1}{\sqrt{2}}, \theta = \frac{\pi}{4}$. Putting everything together, we get:

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{4}} \sin^2(\theta)\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}} 1 - \cos(2\theta)\,d\theta$$

Evaluating this last integral, we get:

$$\frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\cdot 1\right) - 0\right] = \frac{\pi}{8} - \frac{1}{2}$$

(b) Since we see an $\sin^{-1}(x)$, we should think "Integration by Parts". Let

sign		
$+$	$\sin^{-1}(x)$	x
$-$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{2}x^2$

then:

$$\int x\sin^{-1}(x)\,dx = \frac{1}{2}x^2\sin^{-1}(x) - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

That last integral looks very familiar!! Using the integral from the first question,

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) = \frac{1}{2}\theta - \frac{1}{2}\sin(\theta)\cos(\theta)$$

so that:

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\sin^{-1}(x) - \frac{1}{2}x\cdot\sqrt{1-x^2}$$

Altogether, our final answer is then:

$$\frac{1}{2}x^2\sin^{-1}(x) - \frac{1}{4}\sin^{-1}(x) + \frac{1}{4}x\sqrt{1-x^2} + C$$

Alternative Solution (not recommended, but some of you tried it)

Given $\int x\sin^{-1}(x)\,dx$, we could make the substitutions:

$$\theta = \sin^{-1}(x) \quad \sin(\theta) = x \quad \cos(\theta)\,d\theta = dx$$

so we get:

$$\int \sin(\theta)\cdot\theta\cdot\cos(\theta)\,d\theta = \frac{1}{2}\int \theta\sin(2\theta)\,d\theta$$

Now we could do integration by parts:

sign	u	dv
$+$	θ	$\sin(2\theta)$
$-$	1	$-\frac{1}{2}\cos(2\theta)$
$+$	0	$-\frac{1}{4}\sin(2\theta)$

So that:

$$\frac{1}{2}\int \theta \sin(2\theta)\, d\theta = -\frac{1}{4}\theta\cos(2\theta) + \frac{1}{8}\cos(2\theta)$$

Now use trig identities before back substitution:

$$= -\frac{1}{4}\theta\cos(2\theta) + \frac{1}{8}\sin(2\theta) = -\frac{1}{4}\theta\left(1 - 2\sin^2(\theta)\right) + \frac{1}{8}\cdot 2\sin(\theta)\cos(\theta)$$

For back substitution, use a triangle with lengths (adj, opp, hyp)= $(\sqrt{1-x^2}, x, 1)$, respectively:

$$\frac{1}{4}\theta\left(1 - 2\sin^2(\theta)\right) + \frac{1}{8}\cdot 2\sin(\theta)\cos(\theta) = \frac{1}{4}\cdot \sin^{-1}(x)\cdot\left(1 - 2x^2\right) + \frac{1}{4}x\sqrt{1-x^2} + C$$

(c) First, split the integral and then evaluate each. The first one uses a straight u, du substitution, and the second uses an inverse tangent (which can be done right away, or through a triangle).

$$\int \frac{x+2}{x^2+3}\, dx = \int \frac{x}{x^2+3}\, dx + 2\int \frac{1}{x^2+3}\, dx$$

For the first integral, let $u = x^2 + 3$ so that $du = 2x\, dx$:

$$\int \frac{x}{x^2+3}\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+3|$$

For the second integral, we can rewrite:

$$\frac{1}{3\left((x/\sqrt{3})^2 + 1\right)}\, dx$$

Now, $u = x/\sqrt{3}$, $\sqrt{3}\, du = dx$

$$\frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}\, du = \frac{1}{\sqrt{3}}\tan^{-1}(u) = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

Overall, our final answer is:

$$\frac{1}{2}\ln|x^2+3| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

2. Assigned April 10: