FINAL REVIEW FOR CALCULUS II

1. Overarching Theme

The theme of Calc II is: Successive Approximation.

- The integral is approximated by the Riemann Sum. The approximation gets better by using more rectangles. The approximation becomes exact by taking the limit:
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{\infty} f\left(a + i\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n}$$
- The Improper Integral is approximated by a definite integral, and is defined by taking the limit:
$$\int_a^{\infty} f(x)\,dx = \lim_{T\to\infty}\int_a^T f(x)\,dx$$
- Volumes are approximated by slicing and adding the volumes of the shapes together: The Disk, Washer and Cylindrical Shell methods. The method becomes exact by integration.
- An infinite sum is approximated by the finite sum, and is defined by taking the limit:
$$\text{If } S_n = \sum_{k=1}^{n} a_k, \text{ we define } \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n$$
Note that the Riemann Sum is a special case of an infinite series.
- Functions can generally be approximated by polynomials, and the approximation becomes exact when taking a polynomial "of infinite degree". In particular, we had the Taylor and Maclaurin series.

We now turn to an "Executive Summary" of Calculus II. We won't go into too many details here- As you go through the list, be sure you understand the concepts (that is, be sure you go through all your old exams, quizzes, and review sheets).

2. The Integral in Theory

- The definition of the definite integral.
  - Write an integral from a Riemann sum.
  - Write a Riemann sum from an integral.
- The Fundamental Theorem of Calculus, Part I.
  - Sets $g(x) = \displaystyle\int_a^x f(t)\,dt$ as a differentiable function of x.
  - Says that this function is a particular antiderivative of f, $g(a) = 0$.
  - Be able to differentiate:
$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt$$
- The Fundamental Theorem of Calculus, Part II. The main computational tool of Calculus: If F is any antiderivative of the continuous function f,
$$\int_a^b f(x)\,dx = F(b) - F(a)$$
- The Mean Value Theorem for Integrals. The average value of f is attained at some c in $[a,b]$. That is, if f is continuous on $[a,b]$, then there is a c in the interval so that:
$$f_{\text{avg}} = f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Or, the area of the rectangle whose length is $b-a$ and whose height is $f(c)$ is equal to the integral:

$$f(c)(b-a)=\int_a^b f(x)\,dx$$

3. The Integral in Practice

We had several methods to evaluate an integral:

- u, du, or Substitution (Backwards Chain Rule)
- u, dv, or Integration by Parts (Backwards Product Rule) There were three particular types of integrals to remember- $\int x^n g(x)\,dx$, products of exponentials with sines and cosines, and the last type was integrating $\ln(x)$, $\sin^{-1}(x)$, $\tan^{-1}(x)$.
- Partial Fractions. Remember how to integrate something of the form $\int \frac{ax+b}{x^2+c}\,dx$
- Powers of sine and cosine. In particular, remember the formulas for $\sin^2(x)$ and $\cos^2(x)$.
- Trigonometric substitition using triangles. Remember the formula for $\sin(2x)$, and the derivatives of $\tan(x)$, $\sec(x)$.
- Improper Integrals (Integrals where we have to take the limit).

We had several methods to evaluate a volume of revolution: Disks, Washers and Cylindrical Shells.

4. Sequences to Series to Power Series to Taylor Series

Note the evolution of our notation in these sections:

$$\{a_n\}_{n=1}^{\infty}\,,\quad \sum_{k=1}^{\infty} a_k,\quad \sum_{k=1}^{\infty} c_k(x-a)^k,\quad \sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Sequences: What is a sequence? What is the behavior of the sequence as $n\to\infty$? Write a formula for a sequence given the first several terms. Remember how to use l'Hospital's rule.
- Series: What is a series? What does it mean to sum an infinite number of numbers together?
  Template series: Geometric Series, $p-$series, Harmonic series, alternating harmonic series.
  Does the series converge?
  - Check Absolute Convergence first:
    * Ratio Test (Root Test)
    * Comparison (Limit Comparison)
    * Integral Test
  - Check Conditional Convergence last using the Alternating Series Test.
- Power Series: For what values of x does the series converge? What is the radius of convergence? What is the interval of convergence?
  Generally, we will use the Ratio Test to determine the convergence.
  We have one of three choices for convergence. Either the series converges: (i) Only at $x=a$, (ii) for all x, or (iii) for $|x-a|<R$ (and diverges for $|x-a|>R$). In the third case, R is the radius of convergence. Also in the third case, to get the interval of convergence, check your endpoints.
  Template series: $\dfrac{1}{1-x}=\displaystyle\sum_{n=0}^{\infty} x^n$, for $|x|<1$.
  Be able to get new series from the template by differentiation/ integration.
- Taylor Series: Construct a Taylor series approximation to a function. Find the sum of a series by recognizing it as a familiar Taylor series. Recall the Taylor series for e^x, $\sin(x)$, $\cos(x)$, in particular.
  When can a function be represented by it's Taylor Series? Don't all functions do this?