Final Exam Review
Calculus II
Sheet 1

These questions do not include the final section, 11.10. See Sheets 2 and 3 for the types of questions you'll see for this section.

1. State the definition of $\int_a^b f(x)\,dx$. The definition uses the limit of the Riemann sum:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i\cdot\frac{b-a}{n}\right)\cdot\frac{b-a}{n}$$

2. True or False, and give a short reason:

   (a) The Alternating Series Test is sufficient to show that a series is conditionally convergent. This is false. To show that a series converges conditionally, we must first show that is does not converge absolutely, then (normally) use the Alternating Series test to show conditional convergence.

   (b) You can use the Integral Test to show that a series is absolutely convergent. True. The Direct Comparison, Limit Comparison, and Integral tests were all used on positive series.

   (c) Consider $\sum a_n$. If $\lim_{n\to\infty} a_n = 0$, then the sum is said to converge. False. It is not sufficient that the terms of the sum go to zero- although if they don't the series diverges (that was the Divergence Test).

   (d) All continuous functions have antiderivatives. True. This is, in words, what the Fund Theorem of Calc, Part I says in symbols: If f is continuous on $[a, b]$, and $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

   (e) The sequence $a_n = 0.1^n$ converges to $\frac{1}{1-0.1}$
   False. This is a sequence, not a series. The sequence converges to zero.

3. Set up an integral for the volume of the solid obtained by rotating the region defined by $y = \sqrt{x-1}$, $y = 0$ and $x = 5$ about the $y-$axis.

   In this case, the easiest volume method is shells. We see that, for an arbitary shell, the height is $\sqrt{x-1}$ and the radius is x. The volume is then:

$$\int_1^5 2\pi x\sqrt{x-1}\,dx$$

4. Write the area under $y = \sqrt[3]{x}$, $0 \le x \le 8$ as the limit of a Riemann sum (use right endpoints).

   The i^{th} right endpoint is: $0 + i\cdot\frac{8-0}{n} = \frac{8i}{n}$ The i^{th} right height is: $\sqrt[3]{\frac{8i}{n}}$

$$\int_0^8 \sqrt[3]{x}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{\frac{8i}{n}}\cdot\frac{8}{n}$$

5. What is the derivative of e^{-2x}? The antiderivative of e^{-3x}? Same questions for $\sin(3x)$.

$$\frac{d}{dx}\mathrm{e}^{-2x} = -2\mathrm{e}^{-2x}, \qquad \int \mathrm{e}^{-3x}\,dx = \frac{-1}{3}\mathrm{e}^{-3x}.$$

$$\frac{d}{dx}\sin(3x) = 3\cos(3x), \qquad \int \sin(3x)\,dx = \frac{-1}{3}\cos(3x).$$

6. Find $\frac{dy}{dx}$, if $y = \int_{\cos(x)}^{5x} cos(t^2)\, dt$

Remember the general formula from the Fundamental Theorem of Calculus, Part I:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\, dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Therefore, in this particular case,

$$\cos(25x^2)\cdot 5 + \cos\left(\cos^2(x)\right)\cdot \sin(x)$$

7. Let $f(x) = \mathrm{e}^x$ on the interval $[0, 2]$. (a) Find the average value of f. (b) Find c such that $f_{\text{avg}} = f(c)$.

Remember the theorem: If f is continuous on $[a, b]$, then there is a c in $[a, b]$ so that:

$$f_{avg} = f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx$$

So, we first compute the average, then we'll find the c:

$$\frac{1}{2}\int_0^2 \mathrm{e}^x\, dx = \frac{\mathrm{e}^2 - 1}{2}$$

so that $c = \ln\left(\frac{\mathrm{e}^2-1}{2}\right) \approx 1.16$

8. The velocity function is $v(t) = 3t - 5$, $0 \le t \le 3$ (a) Find the displacement. (b) Find the distance traveled.

Remember that the distance traveled is the integral of the absolute value of velocity (displacement is just the integral):

$$\int_0^3 3t - 5\, dt = -\frac{3}{2}$$

$$\int_0^3 |3t-5|\, dt = \int_0^{\frac{5}{3}} -3t + 5\, dt + \int_{\frac{5}{3}}^3 3t - 5\, dt = \frac{41}{6}$$

Does the series converge (absolute or conditional), or diverge?

9. $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$ Note that $\cos(n/2) < 1$ for all n. Therefore, by direct comparison, $\left|\frac{\cos(n/2)}{n^2+4n}\right| < \frac{1}{n^2}$ Thus, the series converges absolutely by the direct comparison test.

10. $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ Using the ratio test, $\lim_{n\to\infty} \frac{(n+1)^2+1}{5^{n+1}} \cdot \frac{5^n}{n^2+1} = \lim_{n\to\infty} \frac{(n+1)^2+1}{n^2+1}\cdot\frac{1}{5} = \frac{1}{5}$ The series converges absolutely by the ratio test.

11. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ By the ratio test, $\lim_{n\to\infty} \frac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n n^2} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 \cdot \frac{3}{n+1} = 0$

Find the interval of convergence:

12. $\sum_{n=1}^{\infty} \frac{n^2 x^n}{10^n}$ (Ratio) $\lim_{n\to\infty} \frac{(n+1)^2|x|^{n+1}}{10^{n+1}}\cdot\frac{10^n}{n^2|x|^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\cdot\frac{|x|}{10} = \frac{|x|}{10}$ The radius of convergence is 10. Check the endpoints:

If $x = 10$, the sum is $\sum n^2$, which diverges. If $x = -10$, the sum is $\sum(-1)^n n^2$, which still diverges. The interval of convergence is therefore: $(-10, 10)$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$ (Ratio) $\displaystyle\lim_{n\to\infty} \frac{|3x-2|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{|3x-2|^n} = \lim_{n\to\infty} \frac{n}{n+1}\frac{|3x-2|}{3} = \frac{|3x-2|}{3}$ From this, the radius of convergence is 3. The interval so far is: $(-1/3, 5/3)$, so now check the endpoints.

If $x = -\frac{1}{3}$, the sum becomes: $\sum \frac{(-1)^n}{n}$, which converges (conditionally). If $x = \frac{5}{3}$, the sum becomes the Harmonic Series, which diverges.

The interval of convergence is $\left[-\frac{1}{3}, \frac{5}{3}\right)$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$

Be careful with the indices on this one. If the n^{th} term has index $2n-1$, then the $n+1^{\text{st}}$ index is $2(n+1)-1 = 2n+1$. Furthermore,

$$\frac{(2n-1)!}{(2n+1)!} = \frac{1\cdot 2\cdot 3\cdots(2n-2)(2n-1)}{1\cdot 2\cdot 3\cdots(2n-2)(2n-1)(2n)(2n+1)} = \frac{1}{(2n)(2n+1)}$$

Now apply the ratio test:

$$\lim_{n\to\infty} \frac{|x|^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{|x|^{2n-1}} = \lim_{n\to\infty} \frac{x^2}{(2n)(2n+1)} = 0$$

for all x. Therefore, the interval of convergence is the set of all real numbers. Evaluate the integral:

15. $\displaystyle\int \frac{1}{y^2-4y-12}\, dy$ (See Ch 7 review, problem 6) Use partial fractions:

$$\frac{1}{y^2-4y-12} = -\frac{1}{8}\cdot\frac{1}{y+2} + \frac{1}{8}\cdot\frac{1}{y-6}$$

so that: $\displaystyle\int \frac{1}{y^2-4y-12}\, dy = -\frac{1}{8}\ln|y+2| + \frac{1}{8}\ln|y-6|$

16. $\displaystyle\int \frac{2}{3x+1} + \frac{2x+3}{x^2+9}\, dx$

The first integral is a straight u, du substitution with $u = 3x+1$. The second integral is something we've seen quite a few times- Remember, split the integral into two:

$$\int \frac{2x}{x^2+9}\, dx + \int \frac{3}{x^2+9}\, dx$$

On the first integral, use u, du substitution with $u = x^2+9$, and on the second integral, use either trig substitution (with triangles), or recall the formula for the inverse tangent.

To use triangles, $\sqrt{x^2+9}$ is the hypotenuse, so that $\tan(\theta) = \frac{x}{3}$, $\sec(\theta) = \frac{\sqrt{x^2-9}}{3}$, and $dx = 3\sec^2(\theta)$. The integral becomes:

$$\int \frac{3}{x^2+9}\, dx = \int \frac{9\sec^2(\theta)}{9\sec^2(\theta)}\, d\theta = \theta = \tan^{-1}(x/3)$$

Altogether, we get:

$$\int \frac{2}{3x+1} + \frac{2x+3}{x^2+9}\, dx = \frac{2}{3}\ln|3x+1| + \ln(x^2+9) + \tan^{-1}(x/3) + C$$

17. $\displaystyle\int x^2\cos(3x)\, dx$ Use integration by parts using a table:

$$\left|\begin{array}{c|c|c} + & x^2 & \cos(3x) \\ - & 2x & \frac{1}{3}\sin(3x) \\ + & 2 & -\frac{1}{9}\cos(3x) \\ - & 0 & -\frac{1}{27}\sin(3x) \end{array}\right| \Rightarrow \int x^2\cos(3x)\, dx = \frac{1}{3}x^2\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + C$$

18. $\int_{-2}^{2} |x-1|\,dx$ Here, we want to split the integral at $x = 1$:

$$\int_{-2}^{2} |x-1|\,dx = \int_{-2}^{1} -x+1\,dx + \int_{1}^{2} x-1\,dx = 5$$

19. $\int \frac{dx}{x\ln(x)}$ Use a u, du substitution with $u = \ln(x)$, $du = \frac{1}{x}\,dx$. This gives:

$$\int \frac{dx}{x\ln(x)} = \int \frac{1}{u}\,du = \ln|u| = \ln(\ln(x)) + C$$

20. $\int x\sqrt{x-1}\,dx$ It's handy to get rid of square roots where possible. Here, try $u = \sqrt{x-1}$ so that $u^2 = x-1$, or $u^2 + 1 = x$. This also gives $2u\,du = dx$. Now,

$$\int x\sqrt{x-1}\,dx = \int (u^2+1)\cdot u \cdot 2u \cdot du = \int 2u^4 + 2u^2\,du = \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$