Final Exam Review
Calculus II
Sheet 2

1. Short Answer:

   (a) Compare and contrast the "Limit Comparison Test" to the "Ratio Test". In both cases, we take the limit of a quotient. In the limit comparison test, the quotient is the ratio of the nth term of the unknown series to the nth term of a known series. In the ratio test, the quotient is the ratio of the n+1st term to the nth term. The conclusions are different- in the limit comparison test, if the limit is any positive, finite number, we say that the two series will converge or diverge together. In the ratio test (since the limit can be thought of as the r in a geo series), if the limit is less than 1, than the series converges (absolutely in both cases). Finally, the ratio test will fail for a $p-$series, so in that case we would have to resort to the comparison test.

   (b) Suppose the power series $\sum c_n x^n$ converges at $x = 3$. What are all the other values for which we know the series must converge? We had a theorem that stated that, for a given series, $\sum c_n (x-a)^n$, there are only three choices for convergence: (i) converges only at $x = a$, (ii) converges for all x, or (iii) converges for $|x-a| < R$, diverges for $|x-a| > R$ (and we have to check endpoints separately).

   (c) If $\sum a_n$, $\sum b_n$ are series with positive terms, and a_n, b_n both go to zero as $n \to \infty$, then what can we conclude if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$?
   We can conclude that the terms of $\sum a_n$ are going to zero faster than b_n. Thus, if $\sum b_n$ is convergent, so is $\sum a_n$, and if $\sum a_n$ is divergent, so is $\sum b_n$.

   (d) What is the derivative of $\sin^{-1}(x)$? Of $\tan^{-1}(x)$? What is the antiderivative of each?
   The derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$. The derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$
   To integrate either, use integration by parts. For $\sin^{-1}(x)$, use $u = \sin^{-1}(x)$, $du = \frac{1}{\sqrt{1-x^2}}$, $dv = dx$, $v = x$:
   $$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$
   For this integral, use $u = 1 - x^2$, $du = -2x\,dx$ to get a final answer:
   $$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

   (e) What was the Mean Value Theorem for Integrals? It said that, if f is continuous on $[a, b]$, then there is a c in $[a, b]$ so that,
   $$f_{\text{avg}} = f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

2. Suppose $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$ and $h''(2) = 13$, and h'' is continuous. Evaluate $\int_1^2 h''(u)\,du$
   $$\int_1^2 h''(u)\,du = h'(2) - h'(1) = 5 - 2 = 3$$

3. Let R be the region in the first quadrant bounded by $y = x^3$ and $y = 2x - x^2$. Calculate: (a) the area of R, (b) Volume by rotating R about the $x-$axis, (c) volume by rotating R about the $y-$axis.

   For part (a), we graph the functions and find the points of intersection, which are $(0, 0)$ and $(1, 1)$. The area of R is then:
   $$\int_0^1 2x - x^2 - x^3\,dx = \frac{5}{12}$$
   For part (b), rotate about the $x-$axis, and using washers,
   $$\int_0^1 \pi\left((2x - x^2)^2 - (x^3)^2\right)\,dx = \frac{41\pi}{105} \approx 1.2267$$

For part (c), rotate about the $y-$axis, and use shells:

$$\int_0^1 2\pi x\left((2x-x^2)-x^3\right)\,dx = \frac{5\pi}{6}$$

4. Find the volume of the solid obtained by rotating the region bounded by $y=\frac{1}{x}$, $y=0, x=1, x=3$ about $y=-1$.

   Draw a picture and using washers, we get:

$$\int_1^3 \pi\left((1+\frac{1}{x})^2-1\right)\,dx = \pi\int_1^3 \frac{2x+1}{x^2}\,dx = \frac{2\pi}{3}+2\pi\ln(3)$$

5. Write the area under $y=\sqrt[3]{x}$, $0\le x\le 8$ as the limit of a Riemann sum (use right endpoints).

   Oops- that's a repeat. See the solutions to sheet 1.

6. Compute $\frac{dg}{dy}$, if $g(y)=\int_3^{\sqrt{y}}\frac{\cos(t)}{t}\,dt$.

   Using the FTC, part I:

$$\frac{\cos(\sqrt{y})}{\sqrt{y}}\cdot\frac{1}{2\sqrt{y}}$$

7. Compute the limit, by using the series for $\sin(x)$: $\lim_{x\to 0}\frac{\sin(x)}{x}$

   (This is from 11.10, so skip it if we haven't talked about it yet) The Maclaurin series for $\sin(x) = x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots$, so dividing this by x gives:

$$\frac{\sin(x)}{x} = 1-\frac{1}{3!}x^2+\frac{1}{5!}x^4+\cdots$$

   so the limit as x approaches 0 is 1.

8. Use the appropriate series to integrate: $\int \mathrm{e}^{x^2}\,dx$

   (This is from 11.10, so skip it if we haven't talked about it yet) The Maclaurin series for $\mathrm{e}^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, so the new series is:

$$\mathrm{e}^{x^2} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$$

   Integrating, we get:

$$\int \mathrm{e}^{x^2} = \sum_{n=0}^{\infty}\frac{1}{n!}\int x^{2n}\,dx = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!(2n+1)}+C$$

Does the given series converge (abs or cond) or diverge?

9. $\sum_{n=1}^{\infty}\left(\frac{3n}{1+8n}\right)^n$

   Looking at this, we want to use the Root Test (since everything is raised to the nth power). The root test says that if $\lim_{n\to\infty}\sqrt[n]{|a_n|}<1$, then the series converges absolutely.

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{3n}{1+8n}\right)^n} = \lim_{n\to\infty}\frac{3n}{8n+1} = \frac{3}{8}<1$$

   Conclusion: The series converges absolutely by the root test.

10. $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{\sqrt{k}}{k+5}$

Before going into details, we see that this sum is quite similar to $\sum(-1)^k \frac{1}{\sqrt{k}}$, which will converge only conditionally. Therefore, we look to the limit comparison to show that the sum does not converge absolutely:

$$\lim_{k\to\infty} \frac{\sqrt{k}}{k+5} \cdot \frac{\sqrt{k}}{1} = \lim_{k\to\infty} \frac{k}{k+5} = 1$$

Therefore, $\sum \frac{\sqrt{k}}{k+5}$, $\sum \frac{1}{\sqrt{k}}$ diverge together.

Note that we could **not** use this direct comparison:

$$\frac{\sqrt{k}}{k+5} < \frac{\sqrt{k}}{k} = \frac{1}{\sqrt{k}}$$

which only tells us that our unknown series is going to zero a bit faster than $\sum \frac{1}{\sqrt{k}}$.

Now we use the Alternating Series Test (show that the positive terms are decreasing to zero):

$$f(x) = \frac{\sqrt{x}}{x+5}, \quad f'(x) = \frac{5-x}{2\sqrt{x}(x+5)^2}$$

so that the derivative is negative for $x > 5$. Now,

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+5} = \lim_{n\to\infty} \frac{1}{\sqrt{n} + \frac{5}{\sqrt{n}}} = 0$$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+\sqrt[3]{n}}{n^2+n^3}$ This looks a lot like a converging $p-$ series, so we do not want to use the ratio test. Instead, let's use a direct comparison:

$$\frac{\sqrt{n}+\sqrt[3]{n}}{n^2+n^3} < \frac{\sqrt{n}+\sqrt[3]{n}}{n^3} = \frac{1}{n^{2.5}} + \frac{1}{n^{7/3}}$$

which forms a sum of converging $p-$series. Therefore, the sum will converge absolutely.

Evaluate, or state that it diverges.

12. $\displaystyle\int e^{-x}\sin(2x)\,dx$. This is the type of integral for which we perform integration by parts twice to get the same integral on both sides of the equation:

$$\left|\begin{array}{c|c|c} + & \sin(2x) & e^{-x} \\ - & 2\cos(2x) & -e^{-x} \\ + & -4\sin(2x) & e^{-x} \end{array}\right| \Rightarrow \int e^{-x}\sin(2x)\,dx = -e^{-x}\sin(2x) - 2e^{-x}\cos(2x) - 4\int e^{-x}\sin(2x)\,dx$$

so that

$$\int e^{-x}\sin(2x)\,dx = -\frac{1}{5}e^{-x}\sin(2x) - \frac{2}{5}e^{-x}\cos(2x)$$

13. $\displaystyle\int \ln(x)\,dx$

Use integration by parts with $u = \ln(x)$, $du = \frac{1}{x}\,dx$, $dv = dx$, $v = x$ to get:

$$\int \ln(x)\,dx = x\ln(x) - \int \frac{x}{x}\,dx = x\ln(x) - x + C$$

14. $\displaystyle\int_0^3 \frac{1}{\sqrt{x}}\,dx$
Note that we have a vertical asymptote at $x = 0$, so

$$\int_0^3 \frac{1}{\sqrt{x}}\,dx = \lim_{T\to 0^+} \int_T^3 x^{-1/2}\,dx = \lim_{T\to 0^+} 2x^{1/2}\Big|_T^3 = 2\sqrt{3} - 0 = 2\sqrt{3}$$

15. $\displaystyle\int \sin^2 x \cos^5 x\,dx$ Recall our rules for dealing with powers of sine and cosine: If both are even, use the formulas for $\sin^2(x)$ and $\cos^2(x)$. If one (or both) are odd, try substitution:

$$\int \sin^2(x)\cos^4(x)\cdot\cos(x)\,dx$$

which means we want to write $u = \sin(x)$. Use the Pythagorean Identity: $\cos^4(x) = (1 - \sin^2(x))^2$, so that:

$$\int \sin^2(x)\cos^4(x)\cdot\cos(x)\,dx = \int \sin^2(x)\left(1 - \sin^2(x)\right)^2\cdot\cos(x)\,dx = -\int u^2(1-u^2)^2\,du$$

Simplify this last integral, and integrate:

$$\int -u^6 + 2u^4 - u^2\,du = \frac{-1}{7}u^7 + \frac{2}{5}u^5 - \frac{1}{3}u^3 + C$$

so our final answer is:

$$\frac{-1}{7}\sin^7(x) + \frac{2}{5}\sin^5(x) - \frac{1}{3}\sin^3(x) + C$$

Find the interval of convergence:

16. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n(\ln(n))^2}$ (Ratio) $\displaystyle\frac{|x|^{n+1}}{(n+1)(\ln(n+1))^2}\cdot\frac{n(\ln(n))^2}{|x|^n} = \left(\frac{n}{n+1}\right)\cdot\left(\frac{\ln(n)}{\ln(n+1)}\right)^2 |x|$
We can break up the limit of a product by taking the product of the limits (if both limits exist), so consider the two expressions separately:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

$$\lim_{n\to\infty}\left(\frac{\ln(n)}{\ln(n+1)}\right)^2 = \left(\lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}\right)^2 = \left(\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}}\right)^2 = 1$$

so the radius of convergence is $|x| < 1$. What about $x = \pm 1$? In that case, we are asking about the convergence of:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

Use the Integral Test for that (we'll get absolute convergence): The terms are positive. The series is decreasing, since $n + 1 > n$ and $\ln(n+1) > \ln(n)$ (directly compare). Now, integrate using a u, du substitution ($u = \ln(x)$),

$$\int_2^{\infty} \frac{1}{x(\ln(x))^2}\,dx = \lim_{T\to\infty}\int_{\ln(2)}^{T} u^{-2}\,du = \lim_{T\to\infty} -u^{-1}\Big|_{\ln(2)}^{T} = \lim_{T\to\infty} -\frac{1}{T} + \frac{1}{\ln(2)} = \frac{1}{\ln(2)}$$

The sum $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^2}$ converges absolutely, and the interval of convergence overall is: $-1 \le x \le 1$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2 2^{2n}}$

Using the ratio test,

$$\frac{|x|^{2n+2}}{((n+1)!)^2 2^{2n+2}} \cdot \frac{(n!)^2 2^{2n}}{|x|^{2n}} = \frac{x^2}{2^2(n+1)^2}$$

The limit of this expression is 0 for all x. The radius of convergence is ∞.

18. Find the radius of convergence: $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n!)} x^n$

Using the ratio test,

$$\frac{((n+1)!)^2 |x|^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2 |x|^n} = \frac{(n+1)^2 |x|}{(2n+1)(2n+2)}$$

The limit of this last expression is: $\dfrac{|x|}{4} < 1$, so the radius of convergence is 4. (Note that this was all the question asked for).