Final Exam Review
Calculus II
Sheet 3

1. True or False, and give a short reason:

   (a) The definition of an infinite sum involves having the terms of the sum go to zero "fast enough".
   False. The definition of an infinite sum said that the sum converges if the partial sums converge. That is, if we define $S_n = \sum_{k=1}^{n} a_k$, then $\sum_{k=1}^{\infty} a_k$ is $\lim_{n\to\infty} S_n$, if that limit exists. We only used this direct definition in some special sums- the geometric series and telescoping series.

   (b) The Ratio Test will not give a conclusive result for $\sum \frac{2n+3}{3n^4+2n^3+3n+5}$
   True. A ratio test will be inconclusive (will return a value of 1) for $p-$like series.

   (c) If $\sum_{n=k}^{\infty} a_n$ converges for some large k, then so will $\sum_{n=1}^{\infty} a_n$.
   True. The first few terms of an infinite sum do not matter- it's what happens as they go to infinity that matters most. We can see this formally: If $\sum_{n=k}^{\infty} a_n$ converges, then:

   $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{k-1} a_n + \sum_{n=k}^{\infty} a_n$$

   This first sum is a sum of a finite number of numbers- it always exists.

   (d) If f is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)\,dx$ converges.
   False: For example, $f(x) = \frac{1}{x+1}$

   (e) Any polynomial with real coefficients can be factored into a product of linear polynomials with real coefficients.
   False But, by our discussions in Partial Fractions, we can factor it into a product of linear factors and irreducible quadratic factors.

   (f) All right cylinders (the wall of the cylinder is at a right angle to the base) whose bases have the same area and whose heights are equal have the same volume.
   True. This is how to remember the formulas for the volumes of disks and washers: The volume is the area of the base $\times$ height.

   (g) If f is continuous and $\displaystyle\int_0^9 f(x)\,dx = 4$, then $\displaystyle\int_0^3 xf(x^2)\,dx = 4$. **False.** Let $u = x^2$, so that

   $$\int_0^3 xf(x^2)\,dx = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}\cdot 4 = 2$$

2. Determine a definite integral representing: $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt{1+\frac{3i}{n}}$ [For extra practice, try writing the integral so that the left endpoint (or bottom bound) must be 5].
Seeing the $\frac{3}{n}$, we take $b-a=3$. Now, since we see $1+\frac{3i}{n}$ we could make a couple of choices:

   - $a = 1$, so that $f(x) = \sqrt{x}$, and $\int_1^4 \sqrt{x}\,dx$
   - $a = 0$, so that $f(x) = \sqrt{1+x}$, and $\int_0^3 \sqrt{1+x}\,dx$
   - $a = 5$, so that $\sqrt{1+\frac{3i}{n}} = \sqrt{-4+(5+\frac{3i}{n}})$, so that $\int_5^8 \sqrt{-4+x}\,dx$

3. Let $\Gamma(n) = \displaystyle\int_0^{\infty} x^{n-1}\mathrm{e}^{-x}\,dx$

   - Show that $\Gamma(1) = 1$: $\Gamma(1) = \displaystyle\int_0^{\infty}\mathrm{e}^{-x}\,dx = \lim_{T\to\infty} -\mathrm{e}^{-x}\Big|_0^T = 0-(-1) = 1$

- Show that $\Gamma(n+1) = n!$, if n is a positive integer (Hint: Integration by parts using a table)

  First, note that if k is a positive integer, then $\frac{x^k}{e^{-x}}\Big|_0^\infty = 0$ Therefore, all of the terms we get (except for the last one) in the table will evaluate to zero. Also, $(-1)^{2n+1} = -1$. This simplifies things to:

$$\begin{array}{lll} + & x^n & e^{-x} \\ - & nx^{n-1} & -e^{-x} \\ + & n(n-1)x^{n-2} & e^{-x} \\ \vdots & \vdots & \vdots \\ (-1)^n & n! & (-1)^n e^{-x} \\ (-1)^{n+1} & 0 & (-1)^{n+1} e^{-x} \end{array} \quad \Rightarrow \int_0^\infty x^n e^{-x}\, dx = \lim_{T\to\infty} -n!\, e^{-x}\Big|_0^T = n!$$

[*This is a good practice problem, but won't be on the exam.*]

4. Evaluate $\int_2^5 (1+2x)\, dx$ by using the definition of the integral. We use equally spaced rectangles with heights at the right endpoints:

$$\int_2^5 (1+2x)\, dx = \lim_{n\to\infty} \sum_{i=1}^n \frac{3}{n}\left(1 + 2\cdot\left[2 + \frac{3i}{n}\right]\right) = \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^n (5 + \frac{6i}{n}) =$$

$$\lim_{n\to\infty} \frac{3}{n}\left(5\sum_{i=1}^n 1 + \frac{6}{n}\sum_{i=1}^n i\right) = \lim_{n\to\infty} \frac{3}{n}\left(5n + \frac{6}{n}\frac{n(n+1)}{2}\right) = \lim_{n\to\infty} 15 + 9\cdot\frac{n+1}{n} = 24$$

5. For each function, find the Taylor series for $f(x)$ centered at the given value of a:

   (a) $f(x) = 1 + x + x^2$ at $a = 2$ We need $f(2), f'(2), f''(2)$: $f(2) = 7$. $f'(x) = 1 + 2x$, so $f'(2) = 5$. $f''(x) = 2$ Now,

$$1 + x + x^2 = 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 = 7 + 5(x-2) + (x-2)^2$$

   (b) $f(x) = \frac{1}{x}$ at $a = 1$. We need to compute derivatives:

n	$f^n(x)$	$f^n(1)$
0	x^{-1}	1
1	$-x^{-2}$	-1
2	$2x^{-3}$	2
3	$-(3\cdot 2)x^{-4}$	$-(3\cdot 2)$
4	$4\cdot 3\cdot 2x^{-5}$	$4\cdot 3\cdot 2$
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n n! x^{-(n+1)}$	$(-1)^n n!$

$$\Rightarrow \frac{f^{(n)}(1)}{n!} = (-1)^n \Rightarrow \sum_{n=0}^\infty (-1)^n (x-1)^n$$

   Alternatively, we could use the geometric series:

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^\infty (1-x)^n = \sum_{n=0}^\infty (-1)^n (x-1)^n$$

6. Find a so that half the area under the curve $y = \frac{1}{x^2}$ lies in the interval $[1, a]$ and half of the area lies in the interval $[a, 4]$. We could set this up multiple ways, but I like to do it like:

$$\int_1^a \frac{1}{x^2}\, dx = \frac{1}{2}\int_1^4 \frac{1}{x^2}\, dx \Rightarrow -\frac{1}{a} + 1 = \frac{3}{8} \Rightarrow a = \frac{8}{5}$$

7. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by $y = x$, $y = 4x - x^2$, about $x = 7$.

   First, find the region of interest. $y = 4x - x^2$ is an upside down parabola with $x-$intercepts at $x = 0, x = 4$. The point of intersection is $x = 4x - x^2 \Rightarrow 0 = 3x - x^2$, or $x = 0$ and $x = 3$. Now the region of interest is between $x = 0$, $x = 3$, above the line $y = x$ and below the parabola $y = 4x - x^2$. Rotate about $x = 7$, and we will use cylindrical shells (Washers would be possible, but messy!). The height of the cylinder is $(4x - x^2) - x = 3x - x^2$. The radius is $7 - x$. Therefore, the integral for the volume is:

$$\int_0^3 2\pi(7 - x)(3x - x^2)\,dx$$

8. Evaluate each of the following: [**The purpose of this problem is to get you to see the differences in notation**]

   (a) $\frac{d}{dx}\int_{3x}^{\sin(x)} t^3\,dt$. By FTC, part I: $\sin^3(x)\cdot\cos(x) - (3x)^3\cdot 3$

   (b) $\frac{d}{dx}\int_1^5 x^3\,dx = 0$ (this is the derivative of a constant)

   (c) $\int_1^5 \frac{d}{dx}x^3\,dx = x^3\big|_1^5 = 5^3 - 1 = 124$. This is FTC, part II.

Converge (absolute or conditional) or Diverge?

9. $\sum_{n=1}^{\infty}\frac{(-1)^n n}{(n+1)(n+2)}$ This will behave like $\sum(-1)^n\frac{1}{n}$, which only converges conditionally. We cannot use the Ratio Test, so we show that it does NOT converge absolutely by comparing it to $\frac{1}{n}$:

$$\lim_{n\to\infty}\frac{n}{(n+1)(n+2)}\cdot\frac{n}{1} = 1$$

   Therefore, the series does not converge absolutely. Now we use the Alternating Series Test to show that it converges conditionally: Each term is clearly positive, for $n > 0$. Is it decreasing?

$$f(x) = \frac{x}{(x+1)(x+2)} \qquad f'(x) = \frac{2 - x^2}{(x+1)^2(x+2)^2}$$

   so the derivative is negative for $x > \sqrt{2}$ (or the terms of the series are decreasing for $n > 2$). Finally, show that the terms are going to zero:

$$\lim_{n\to}\frac{n}{(n+1)(n+2)} = \lim_{n\to\infty}\frac{n}{n^2+3n+2} = \lim_{n\to\infty}\frac{1}{2n+3} = 0$$

   (the last equality by l'Hospital's rule).

10. $\sum_{n=1}^{\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ This one is very similar to a take home quiz problem, so we won't go into many details. It will converge by comparing it to $\frac{1}{n^2}$.

11. $\sum_{k=1}^{\infty}\frac{4^k + k}{k!}$ Use the ratio test:

$$\frac{4^{k+1} + (k+1)}{(k+1)!}\cdot\frac{k!}{4^k + k} = \frac{4^{k+1} + k + 1}{(k+1)(4^k + k)} = \frac{4 + \frac{k}{4^k} + \frac{1}{4^k}}{(k+1)(1 + \frac{k}{4^k})}$$

   The numerator approaches 4 as $k \to \infty$ and the denominator goes to ∞ as $k \to \infty$, so overall, the limit is 0. Therefore, this series converges (absolutely) by the Ratio Test.

12. $\sum_{n=1}^{\infty} n^n x^n$ By the root test, $\lim_{n\to\infty}(n^n x^n)^{1/n} = \lim_{n\to\infty} nx = \infty$ Therefore, the only point of convergence is when $x = 0$.

13. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$ (Ratio) $\lim_{n\to\infty} \frac{|x+2|^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n4^n}{|x|^n} = \lim_{n\to\infty} \frac{n+1}{n}\frac{|x+2|}{4} = \frac{|x+2|}{4}$

Check the endpoints: If $x = 2$, then the sum is $\sum \frac{1}{n}$ which diverges. If $x = -6$, then the sum is $\sum \frac{(-1)^n}{n}$, which converges. The interval of convergence is therefor $-6 \le x < 2$.

14. $\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$ (Ratio) $\lim_{n\to\infty} \frac{2^{n+1}|x-3|^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{2^n|x-3|^n} = \lim_{n\to\infty} \sqrt{\frac{n+3}{n+4}} \cdot 2|x-3| = 2|x-3|$ So far, $\frac{5}{2} < x < \frac{7}{2}$. Now check endpoints:

If $x = \frac{5}{2}$, the sum becomes $\sum \frac{(-1)^n}{\sqrt{n+3}}$, which converges by the Alternating Series test, and if $x = \frac{7}{2}$, the sum becomes $\sum \frac{1}{\sqrt{n+3}}$ which diverges (p-series).

15. $\int_0^{\infty} \frac{1}{(x+2)(x+3)}\,dx$ By partial fractions, $\int \frac{1}{(x+2)(x+3)}\,dx = \int \frac{1}{x+2} - \frac{1}{x+3}\,dx = \ln|x+2| - \ln|x+3| = \ln\left|\frac{x+2}{x+3}\right|$ As $x \to \infty$, $\ln\left|\frac{x+2}{x+3}\right| \to \ln(1) = 0$. Altogether we get:

$$\int_0^{\infty} \frac{1}{(x+2)(x+3)}\,dx = 0 - \ln(2/3) = \ln(3/2)$$

16. $\int u(\sqrt{u} + \sqrt[3]{u})\,du$ Simplify algebraically first, to get $\int u^{3/2} + u^{4/3}\,du = \frac{2}{5}u^{5/2} + \frac{3}{7}u^{7/3} + C$

17. $\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$ Use a triangle whose hypotenuse is 2, side opposite θ is x, and side adjacent is $\sqrt{4-x^2}$. Then, substitute $2\sin(\theta) = x$, $2\cos(\theta) = \sqrt{4-x^2}$, and we get:

$$\int \frac{4\sin^2(\theta)\cdot 2\cos(\theta)}{2^3\cos^3(\theta)}\,d\theta = \int \tan^2(\theta)\,d\theta = \int \sec^2(\theta) - 1\,d\theta = \tan(\theta) - \theta$$

Convert back using triangles to get: $\frac{x}{\sqrt{4-x^2}} - \sin^{-1}(x/2) + C$

18. $\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$ Let $u = \tan^{-1}(x)$, so $du = \frac{1}{1+x^2}\,dx$. Then the integral becomes $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\tan^{-1}(x))^2 + C$

19. $\int \frac{1}{\sqrt{x^2-4x}}\,dx$ "Complete the Square" in the denominator to get $x^2 - 4x = (x-2)^2 - 4$. Now, use a triangle whose hypotenuse is $x-2$, side adjacent is 2, and side opposite is $\sqrt{(x-2)^2 - 2^2}$. Then,

$$2\tan(\theta) = \sqrt{(x-2)^2 - 2^2}, \quad 2\sec(\theta) = x - 2, \quad 2\sec(\theta)\tan(\theta)d\theta = dx$$

Substituting, we get:

$$\int \frac{1}{\sqrt{x^2-4x}}\,dx = \int \frac{2\sec(\theta)\tan(\theta)}{2\tan(\theta)}\,d\theta = \int \sec(\theta)\,d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$$

[NOTE: You'll be given the formulas as in Exam 2]. Final answer: $\ln\left|\frac{x-2}{2} + \frac{\sqrt{(x-2)^2-4}}{2}\right| + C$

20. $\int x^4 \ln(x)\,dx$ Use integration by parts with $u = \ln(x)$, $dv = x^4\,dx$ to get:

$$\int x^4 \ln(x)\,dx = \frac{1}{5}x^5\ln(x) - \frac{1}{25}x^5 + C$$