Review: Limits at Infinity

These are the techniques we had from Calculus I to find the horizontal asymptote of a function.

1. When dealing with a fraction:
   (a) "$\frac{k}{\infty} = 0$" If the numerator goes to a constant, and the denominator goes to infinity, then overall the fraction goes to zero.
   (b) "$\frac{k}{0}$ = DNE" If the numerator goes to a constant, and the denominator goes to 0, then overall the fraction goes to infinity (plus or minus); or you can say that it Does Not Exist (DNE).
   (c) Standard Algebraic Techniques:
      i. Divide by a power of x (RECALL: If $x > 0$, $x = \sqrt{x^2}$. If $x < 0$, then $x = -\sqrt{x^2}$)
      ii. Rationalize (usually when square roots are added/subtracted)
   (d) Some examples:

$$\lim_{x\to\infty} \frac{x^2 - x}{3x^2 - 4} = \lim_{x\to\infty} \frac{1 - \frac{x}{x^2}}{3 - 4\frac{4}{x^2}} = \frac{1-0}{3-0} = \frac{1}{3}$$

$$\lim_{x\to\infty} \sqrt{x+3} - \sqrt{x} = \lim_{x\to\infty} \sqrt{x+3} - \sqrt{x} \cdot \frac{\sqrt{x+3} + \sqrt{x}}{\sqrt{x+3} + \sqrt{x}} =$$

$$\lim_{x\to\infty} \frac{x+3-x}{\sqrt{x+3} + \sqrt{x}} = 0$$

2. Some functions with horizontal asymptotes:
   (a) $f(x) = \tan^{-1}(x)$ has horizontal asymptotes at $y = \frac{\pi}{2}$ (as $x \to \infty$), and $\frac{-\pi}{2}$ (as $x \to -\infty$)
   (b) $f(x) = \frac{1}{x^p}$ has a horizontal asymptote at $y = 0$ if $p \geq 1$ and $x \to \infty$.

3. Using l'Hospital's Rule:

   The Rule: If you have "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$", then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

   You may not be given a fraction- we may have to do some algebra first! Sometimes you can use l'Hospital's rule for a product, $f(x) \cdot g(x)$, and sometimes for exponentiation, $(f(x))^{g(x)}$. See the examples below.

   NOTE: $f(x)^{g(x)} = \mathrm{e}^{g(x)\cdot\ln(f(x))}$

4. Examples

(a) $\lim_{x\to\infty} x^{\frac{1}{x}}$

First, some algebra: $x^{\frac{1}{x}} = \mathrm{e}^{\frac{1}{x}\ln(x)}$

Now, we need the limit of $\frac{\ln(x)}{x}$, which we can get using l'Hospital's rule:

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

so overall:

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} \mathrm{e}^{\frac{1}{x}\ln(x)} = \mathrm{e}^{\lim_{x\to\infty} \frac{\ln(x)}{x}} = \mathrm{e}^0 = 1$$

(b) $\lim_{x\to\infty} x^2\mathrm{e}^{-x}$

Rewrite the product as a fraction so we can apply l'Hospital's rule:

$$\lim_{x\to\infty} x^2\mathrm{e}^{-x} = \lim_{x\to\infty} \frac{x^2}{\mathrm{e}^x} = \lim_{x\to\infty} \frac{2x}{\mathrm{e}^x} = \lim_{x\to\infty} \frac{2}{\mathrm{e}^x} = 0$$

(c) $\lim_{x\to\infty} \ln(x+1) - \ln(x)$

Rewrite using the rules of logarithms:

$$\lim_{x\to\infty} \ln(x+1)-\ln(x) = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = \ln\left(\lim_{x\to\infty} \frac{x+1}{x}\right) = \ln(1) = 0$$

(d) $\lim_{x\to\infty} (1+1/x)^x$

Rewrite using the exponential function:

$$(1+1/x)^x = \mathrm{e}^{x\ln(1+1/x)}$$

Then take the limit of this exponent:

$$\lim_{x\to\infty} x\ln(1+1/x) = \lim_{x\to\infty} \frac{\ln(1+1/x)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = 1$$

so overall,

$$\lim_{x\to\infty} (1+1/x)^x = \lim_{x\to\infty} \mathrm{e}^{x\ln(1+1/x)} = \mathrm{e}^1 = \mathrm{e}$$

(e) $\lim_{x\to-\infty} \frac{\sqrt{x^2+3}}{3x-2}$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+3}}{3x-2} \cdot \frac{-\frac{1}{\sqrt{x^2}}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{3}{x^2}}}{3-\frac{2}{x}} = -\frac{1}{3}$$