Take Home Quiz Solutions

Test the series for convergence or divergence. If the series converges, say whether it is absolute or conditional. Be specific about your reasons!

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n-1}{n^2+n}$

   We might see it right off the bat, but this looks a lot like the alternating harmonic series- which converges only conditionally. To see this, we can first show that it does not converge absolutely by comparing it to the harmonic series (the limit comparison test):

   $$\lim_{n\to\infty}\frac{b_n}{a_n}=\lim_{n\to\infty}\frac{\frac{n-1}{n^2+n}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^2-n}{n^2+n}=1$$

   Since $\sum\frac{1}{n}$ diverges, so does $\sum\frac{n-1}{n^2+n}$.

   Now we can apply the Alternating Series Test:

   - The terms $\frac{n-1}{n^2+n}$ are positive for $n>1$.
   - The terms are decreasing: After simplification, the derivative is:

     $$f'(x)=\frac{x^2+x-(x-1)(2x+1)}{(x^2+x)^2}=-\frac{x^2-2x-1}{x^2(x+1)^2}$$

     The numerator, x^2-2x-1 is positive if $x>1+\sqrt{2}\approx 2.4$, so the expression overall will be negative if $x>3$. Therefore, the function will decreasing.

     Alternatively, we can write the numerator as:

     $$-x^2+2x+1=-(x^2-2x+\quad)+1=-(x^2-2x+1)+1+1=-(x-1)^2+2$$

     Which is negative for $x>1+\sqrt{2}$
   - The terms are going to zero:

     $$\lim_{n\to\infty}\frac{n-1}{n^2+n}=\lim_{n\to\infty}\frac{1-\frac{1}{n}}{n+1}=0$$

2. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{10^n}{n!}$

   Use the ratio test to see if this converges absolutely:

   $$\lim_{n\to\infty}\frac{10^{n+1}}{(n+1)!}\cdot\frac{n!}{10^n}=\lim_{n\to\infty}\frac{10}{n+1}=0$$

   Therefore, the series converges, and it converges absolutely.

3. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{2\cdot 5\cdot 8\cdots(3n+2)}$

   Try the ratio test:

   $$\lim_{n\to\infty}\frac{(n+1)!}{2\cdot 5\cdots(3n+2)(3n+5)}\cdot\frac{2\cdot 5\cdot 8\cdots(3n+2)}{n!}=\lim_{n\to\infty}\frac{n+1}{3n+5}=\frac{1}{3}$$

   Therefore, the series converges (absolutely).

4. $\displaystyle\sum_{n=1}^{\infty}n^2\mathrm{e}^{-n^3}$

   With these terms, we might go immediately to the integral test, with $f(x)=x^2\mathrm{e}^{-x^3}$

- Is the function positive? Yes.
- Is the function decreasing? Differentiate and factor:

$$f'(x) = x(2 - 3x^3)\mathrm{e}^{-x^3}$$

  Where is this negative? e^{-x^3} is always positive, and $x > 0$. The derivative will be negative ($x > 0$) if $2 - 3x^3 < 0$. This will be true if $x > \sqrt[3]{\frac{2}{3}} \approx 0.8$.
- Now integrate:

$$\int_1^{\infty} x^2 \mathrm{e}^{-x^3}\, dx = \frac{1}{3}\int_1^{\infty} \mathrm{e}^{-u}\, du = -\frac{1}{3}\mathrm{e}^{-u}\Big|_1^{\infty} = \frac{1}{3}\mathrm{e}^{-1}$$

  Therefore, the area under $f(x)$ is finite, so the series will converge (absolutely, since the expression is always positive anyway).

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

   This looks like a series with $\frac{n}{n^3} = \frac{1}{n^2}$, so we expect it to converge (absolutely, since the terms are all positive).

   We use the limit comparison test, since the ratio test will be inconclusive:

$$\lim_{n\to\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5} \cdot \frac{n^2}{1}$$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$

   Use the root test:

$$\lim_{n\to\infty} \sqrt[n]{\frac{(2n)^n}{n^{2n}}} = \lim_{n\to\infty} \frac{2n}{n^2} = \lim_{n\to\infty} \frac{2}{n} = 0$$

   Therefore, the series converges absolutely.

7. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

   We might see immediately that this series will not converge absolutely, since it is like $\sum \frac{1}{\sqrt{n}}$

   To show this, let's go ahead and compare it:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} \cdot \frac{\sqrt{n}}{1} = 1$$

   So this series diverges by the limit comparison test.

   On the other hand, we can show that it converges using the Alternating Series Test:

   - The terms $\frac{1}{\sqrt{n}-1}$ are positive.
   - They are decreasing:

$$\frac{1}{\sqrt{n+1}-1} \leq \frac{1}{\sqrt{n}-1}$$

   - And they are going to zero:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$$

   Conclusion: The series $\sum \frac{(-1)^n}{\sqrt{n}-1}$ converges, but only conditionally.