Solutions to Review Questions, 5.3-5.5

1. True or False, and give a short reason:

   (a) If f and g are continuous on $[a,b]$, then $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

   True. This is one of the properties of definite integrals.

   (b) If f and g are continuous on $[a,b]$, then $\int_a^b f(x)g(x)\,dx = \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$

   Oooh- this is bad on so many levels! We can use just about any functions f and g and this will not work. For example,

$$\int x\cdot x\,dx = \frac{1}{3}x^3 + C$$

   but $\int x\,dx \cdot \int x\,dx = \left(\frac{1}{2}x^2 + C_1\right)\left(\frac{1}{2}x^2 + C_2\right)$

   Another explanation would be that since

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) \neq f'(x)\cdot g'(x)$$

   the reverse does not work, either.

   (c) If f is continuous on $[a,b]$, then $\int_a^b xf(x)\,dx = x\int_a^b f(x)\,dx$

   False. The result of doing this would give you a function of x, but the definite integral is a number. It is also false in general, $\int xf(x)\,dx \neq x\int f(x)\,dx$; for example $\int x\,dx = \frac{1}{2}x^2 + C$, but $x\int\,dx = x^2 + Cx$

   (d) If f' is continuous on $[-1,4]$, then $\int_{-1}^4 f'(w)\,dw = f(4) - f(-1)$

   This is true, since the antiderivative of f' is f (that is, the derivative of f is f'). So, this is the Fundamental Theorem of Calculus, part II.

   (e) $\int_{-2}^1 \frac{1}{x^4}\,dx = -\frac{3}{8}$

   False. We cannot use the FTC on this integral since $\frac{1}{x^4}$ has a vertical asymptote at $x=0$, which is inside the interval $[-2,1]$.

   (f) All continuous functions have derivatives.

   False. For example, $y = |x|$ does not have a derivative at $x=0$, although it is continuous there.

   (g) All continuous functions have antiderivatives.

   True. This is the Fundamental Theorem of Calculus, Part I written out in words.

   (h) If $v(t)$ is velocity at time t, then the distance traveled between times 3 and 7 is given by $\int_3^7 v(t)\,dt$

   False. To get the total distance traveled, we would evaluate $\int_3^7 |v(t)|\,dt$. If we want the *relative* distance, we would ask for *displacement*, which is given as $\int_3^7 v(t)\,dt$

   (i) Even though the function:

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 3+x & \text{if } x > 1 \end{cases}$$

   is not continuous at $x=1$, we can compute $\int_0^2 f(x)\,dx$.

   This is true. Because there is not a vertical asymptote at $x=1$, we can compute the integral by breaking it up:

$$\int_0^2 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 3+x\,dx$$

2. Compare the notation:

   (a) $\frac{d}{dx}\int_a^x f(t)\,dt$ This value is $f(x)$

   (b) $\frac{d}{dx}\int_a^b f(t)\,dt$ This value is zero (it's the derivative of a constant).

   (c) $\int_a^b \frac{d}{dx} f(x)\,dx$ This value is $f(b)-f(a)$

   (d) $\int_a^b f(x)\,dt$ This is a little tricky- since the integral is in terms of t, but f is an expression in x, it will be constant with respect to t. The answer: $f(x)(b-a)$.

3. Evaluate, where possible. If not, state why:

   (a) $\int_1^4 \frac{x^2-x+1}{\sqrt{x}}\,dx = \int_1^4 x^{3/2}-x^{1/2}+x^{-1/2}\,dx = \frac{2}{5}x^{5/2}-\frac{2}{3}x^{3/2}+2x^{1/2}\Big|_1^4 = \frac{146}{15}$

   (b) $\int_0^2 \frac{x}{(x^2-1)^2}\,dx$ The function has a vertical asymptote at $x=1$; FTC does not apply.

   (c) $\frac{d}{dx}\int_3^{3^x} t\,dt = 3^x\cdot 3^x\ln(3) = 3^{2x}\ln(3)$

   (d) $\int (1-x)\sqrt{2x-x^2}\,dx$ Use $u=2x-x^2$, $du = 2-2x\,dx$, or $\frac{1}{2}du = 1-x\,dx$. This gives:

$$\int (1-x)\sqrt{2x-x^2}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}+C = \frac{1}{3}\cdot(2x-x^2)^{3/2}+C$$

   (e) $\int \frac{\cos(\ln(x))}{x}\,dx$ Use $u=\ln(x)$, so $du = \frac{1}{x}\,dx$, which gives:

$$\int \frac{\cos(\ln(x))}{x}\,dx = \int \cos(u)\,du = \sin(u)+C = \sin(\ln(x))+C$$

   (f) $\int_0^1 \frac{d}{dx}\left(\frac{e^x}{x+1}\right)\,dx = \frac{e^x}{x+1}\Big|_0^1 = \frac{1}{2}e-1$

   (g) $\int_0^{2\pi} |\sin(x)|\,dx$ The sine function is positive for $0<x<\pi$ and negative for $\pi<x<2\pi$, so we rewrite:

$$\int_0^{\pi}\sin(x)\,dx - \int_{\pi}^{2\pi} = -\cos(x)|_0^{\pi} + \cos(x)|_{\pi}^{2\pi} = 2+2=4$$

   (h) $\int \frac{x}{\sqrt{1-x^4}}\,dx$ You might have tried $u=1-x^4$, but that doesn't get us very far. Note that this expression is close to $\frac{1}{\sqrt{1-x^2}}$, whose antiderivative is $\sin^{-1}(x)$. This suggests that we try: $u=x^2$, $du=2x\,dx$ or $\frac{1}{2}du = x\,dx$. Putting this together,

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\sin^{-1}(u)+C = \frac{1}{2}\sin^{-1}(x^2)+C$$

   (i) $\frac{d}{dx}\int_{2x}^{3x+1}\sin(t^4)\,dt$ Here we use the general formula: $\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x)$. First, rewrite:

$$\frac{d}{dx}\int_{2x}^{3x+1}\sin(t^4)\,dt = \frac{d}{dx}\int_{2x}^{1}\sin(t^4)\,dt + \frac{d}{dx}\int_1^{3x+1}\sin(t^4)\,dt =$$

$$-\frac{d}{dx}\int_1^{2x}\sin(t^4)\,dt + \frac{d}{dx}\int_1^{3x+1}\sin(t^4)\,dt = -\sin((2x)^4)\cdot 2+\sin((3x+1)^4)\cdot 3$$

(j) $\int \frac{x^2}{\sqrt{1-x}}\,dx$ Here, we'll try $u = 1-x$, so that $du = -dx$. But we still have x^2 in the numerator. Use $u = 1-x$ so that $x = 1-u$ and $x^2 = (1-u)^2 = 1-2u+u^2$, and now do the full substitution:

$$\int \frac{x^2}{\sqrt{1-x}}\,dx = -\int \frac{1-2u+u^2}{\sqrt{u}}\,du = -\int (u^{-1/2} - 2u^{1/2} + u^{3/2})\,du =$$

$$-\left(2u^{1/2} - 2\cdot\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right) + C = -2\sqrt{1-x} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C$$

Note: Since C is arbitrary, you could have either C or $-C$ in your answer

4. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$ With that $2x$, we should be tempted to use $u = 2x$, $du = 2\,dx$ or $\frac{1}{2}du = dx$. Now, for the bounds, if $x = 0$, then $u = 0$ and if $x = 2$, $u = 4$. This says:

$$\int_0^2 f(2x)\,dx = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}\cdot 10 = 5$$

5. If $g(x) = \int_0^x \frac{1}{1+t+t^2}\,dt$, find where g is concave up. To find where g is concave up, we must compute its second derivative and find where it is positive:

$$g'(x) = \frac{1}{1+x+x^2} = (1+x+x^2)^{-1} \qquad g''(x) = \frac{-(1+2x)}{(1+x+x^2)^2}$$

Since the denominator is always positive, we need look only for where the numerator is positive:

$$-1-2x > 0 \Rightarrow -2x > 1 \Rightarrow x < \frac{1}{2}$$

If $x < \frac{1}{2}$, then $g''(x) > 0$, so g is concave up.

6. If $\int_0^6 f(x)\,dx = 10$ and $\int_0^4 f(x)\,dx = 7$, find $\int_4^6 f(x)\,dx$. By a property of integrals,

$$\int_4^6 f(x)\,dx = \int_0^6 f(x)\,dx - \int_0^4 f(x)\,dx = 10 - 7 = 3$$

Challenge Problems!

If you breezed through the previous questions, and are looking for something more challenging, try these!

1.

$$\frac{d^2}{dx^2}\int_0^x \left(\int_1^{\sin(t)} \sqrt{1+u^4}\,du\right)dt$$

First, let $F(t) = \int_1^{\sin(t)} \sqrt{1+u^4}\,du$, so that the first derivative is:

$$\frac{d}{dx}\int_0^x F(t)\,dt = F(x) = \int_1^{\sin(x)} \sqrt{1+u^4}\,du$$

Now the next derivative is our standard problem:

$$\frac{d}{dx}\int_1^{\sin(x)} \sqrt{1+u^4}\,du = \sqrt{1+(\sin(x))^4}\cdot\cos(x)$$

2. If f is a differentiable function so that: $\displaystyle\int_0^x f(t)\,dt = (f(x))^2$ for all x, find f.

   We can remove the integral by differentiating both sides:

$$f(x) = 2f(x)\cdot f'(x) \Rightarrow f(x) - 2f(x)\cdot f'(x) = 0 \Rightarrow f(x)\,(1-2f'(x)) = 0$$

   This says that either $f(x) = 0$ or $1 - 2f'(x) = 0$. For the first case, we see that $f(x) = 0$ will solve our original function, since $\int_0^x 0\,dx = 0$ for all x.

   In the second case, $f'(x) = \frac{1}{2}$, so $f(x) = \frac{1}{2}x + C$. To get the value of C, notice in the original equation that if $x = 0$, then:

$$\int_0^0 f(x)\,dx = (f(0))^2 \Rightarrow f(0) = 0$$

   Thus, $C = 0$.

   So, we have two possibilities: $f(x) = 0$ or $f(x) = \frac{1}{2}x$.

3. Find

$$\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt$$

   Do we recognize this as a derivative? If $g(x) = \displaystyle\int_0^x f(t)\,dt$, then

$$g'(2) = \lim_{h\to 0}\frac{1}{h}(g(2+h)-g(2)) = \lim_{h\to 0}\frac{1}{h}\left(\int_0^{2+h} f(t)\,dt - \int_0^2 f(t)\,dt\right) = \lim_{h\to 0}\frac{1}{h}\int_2^{2+h} f(t)\,dt$$

   In particular, if $g(x) = \displaystyle\int_0^x \sqrt{1+t^3}\,dt$, then $g'(x) = \sqrt{1+t^3}$, so

$$\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt = g'(2) = \sqrt{1+2^3} = \sqrt{9} = 3$$

   Note: the use of 0 in the definition of g was arbitrary- we could've used any constant greater than -1.