Solutions to Review Questions, 5.3-5.5

1. True or False, and give a short reason:

(a) If \( f \) and \( g \) are continuous on \([a, b]\), then
\[
\int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx
\]
True. This is one of the properties of definite integrals.

(b) If \( f \) and \( g \) are continuous on \([a, b]\), then
\[
\int_a^b f(x)g(x) \, dx = \int_a^b f(x) \, dx \cdot \int_a^b g(x) \, dx
\]
Oooh- this is bad on so many levels! We can use just about any functions \( f \) and \( g \) and this will not work. For example,
\[
\int_0^1 x \cdot x \, dx = \frac{1}{3} x^3 + C
\]
but
\[
\int_0^1 x \, dx \cdot \int_0^1 x \, dx = \left( \frac{1}{2} x^2 + C_1 \right) \left( \frac{1}{2} x^2 + C_2 \right)
\]
Another explanation would be that since
\[
\frac{d}{dx} (f(x) \cdot g(x)) \neq f'(x) \cdot g'(x)
\]
the reverse does not work, either.

(c) If \( f \) is continuous on \([a, b]\), then
\[
\int_a^b xf(x) \, dx = x \int_a^b f(x) \, dx
\]
False. The result of doing this would give you a function of \( x \), but the definite integral is a number. It is also false in general, \( \int_a^b xf(x) \, dx \neq x \int_a^b f(x) \, dx \); for example \( \int_1^2 x \, dx = \frac{1}{2} x^2 + C \), but \( x \int_1^2 x \, dx = x^2 + C x \)

(d) If \( f' \) is continuous on \([-1, 4]\), then
\[
\int_{-1}^{4} f'(w) \, dw = f(4) - f(-1)
\]
This is true, since the antiderivative of \( f' \) is \( f \) (that is, the derivative of \( f \) is \( f' \)). So, this is the Fundamental Theorem of Calculus, part II.

(e) \[
\int_{-2}^{1} \frac{1}{x^4} \, dx = -\frac{3}{8}
\]
False. We cannot use the FTC on this integral since \( \frac{1}{x^4} \) has a vertical asymptote at \( x = 0 \), which is inside the interval \([-2, 1]\).

(f) All continuous functions have derivatives.
False. For example, \( y = |x| \) does not have a derivative at \( x = 0 \), although it is continuous there.

(g) All continuous functions have antiderivatives.
True. This is the Fundamental Theorem of Calculus, Part I written out in words.

(h) If \( v(t) \) is velocity at time \( t \), then the distance traveled between times 3 and 7 is given by \( \int_3^7 v(t) \, dt \)
False. To get the total distance traveled, we would evaluate \( \int_3^7 |v(t)| \, dt \). If we want the relative distance, we would ask for displacement, which is given as \( \int_3^7 v(t) \, dt \)

(i) Even though the function:
\[
f(x) = \begin{cases} 
  x^2 & \text{if } x < 1 \\
  3 + x & \text{if } x > 1
\end{cases}
\]
is not continuous at \( x = 1 \), we can compute \( \int_0^2 f(x) \, dx \).
This is true. Because there is not a vertical asymptote at \( x = 1 \), we can compute the integral by breaking it up:
\[
\int_0^2 f(x) \, dx = \int_0^1 x^2 \, dx + \int_1^2 3 + x \, dx
\]
2. Compare the notation:

(a) \( \frac{d}{dx} \int_a^x f(t) \, dt \) This value is \( f(x) \)

(b) \( \frac{d}{dx} \int_a^b f(t) \, dt \) This value is zero (it’s the derivative of a constant).

(c) \( \int_a^b \frac{d}{dx} f(x) \, dx \) This value is \( f(b) - f(a) \)

(d) \( \int_a^b f(x) \, dx \) This is a little tricky- since the integral is in terms of \( t \), but \( f \) is an expression in \( x \), it will be constant with respect to \( t \). The answer: \( f(x)(b - a) \).

3. Evaluate, where possible. If not, state why:

(a) \[ \int_1^4 \frac{x^2 - x + 1}{\sqrt{x}} \, dx = \int_1^4 x^{3/2} - x^{1/2} + x^{-1/2} \, dx = \frac{2}{5} x^{5/2} - \frac{2}{3} x^{3/2} + 2x^{1/2} \bigg|_1^4 = \frac{146}{15} \]

(b) \[ \int_0^2 \frac{x}{(x^2 - 1)^2} \, dx \] The function has a vertical asymptote at \( x = 1 \); FTC does not apply.

(c) \[ \frac{d}{dx} \int_3^{x^3} t \, dt = 3x^2 \cdot 3^x \ln(3) = 3^{2x} \ln(3) \]

(d) \( \int (1 - x)\sqrt{2x - x^2} \, dx \) Use \( u = 2x - x^2, \, du = 2 - 2x \, dx \), or \( \frac{1}{2} \, du = 1 - x \, dx \). This gives:

\[ \int (1 - x)\sqrt{2x - x^2} \, dx = \frac{1}{2} \int u^{1/2} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{1}{3} \cdot (2x - x^2)^{3/2} + C \]

(e) \[ \int \frac{\cos(\ln(x))}{x} \, dx \] Use \( u = \ln(x) \), so \( du = \frac{1}{x} \, dx \), which gives:

\[ \int \frac{\cos(\ln(x))}{x} \, dx = \int \cos(u) \, du = \sin(u) + C = \sin(\ln(x)) + C \]

(f) \[ \int_0^1 \frac{e^x}{x + 1} \, dx = \frac{e^x}{x + 1}\bigg|_0^1 = \frac{1}{2} \cdot e - 1 \]

(g) \[ \int_0^\pi |\sin(x)| \, dx \] The sine function is positive for \( 0 < x < \pi \) and negative for \( \pi < x < 2\pi \), so we rewrite:

\[ \int_0^\pi \sin(x) \, dx - \int_\pi^{2\pi} = -\cos(x)|_0^\pi + \cos(x)|_{2\pi} = 2 + 2 = 4 \]

(h) \[ \int \frac{x}{\sqrt{1 - x^4}} \, dx \] You might have tried \( u = 1 - x^4 \), but that doesn’t get us very far. Note that this expression is close to \( \frac{1}{\sqrt{1-x}} \), whose antiderivative is \( \sin^{-1}(x) \). This suggests that we try: \( u = x^2, \, du = 2x \, dx \) or \( \frac{1}{2} \, du = x \, dx \). Putting this together,

\[ \int \frac{x}{\sqrt{1 - x^4}} \, dx = \frac{1}{2} \int \frac{1}{\sqrt{1 - u^2}} \, du = \frac{1}{2} \sin^{-1}(u) + C = \frac{1}{2} \sin^{-1}(x^2) + C \]

(i) \[ \frac{d}{dx} \int_{2x}^{3x+1} \sin(t^4) \, dt \] Here we use the general formula: \( \frac{d}{dx} \int_a^b f(t) \, dt = f(h(x)) \cdot h'(x) \). First, rewrite:

\[ \frac{d}{dx} \int_{2x}^{3x+1} \sin(t^4) \, dt = \frac{d}{dx} \int_2^1 \sin(t^4) \, dt + \frac{d}{dx} \int_1^{3x+1} \sin(t^4) \, dt = -\frac{d}{dx} \int_2^1 \sin(t^4) \, dt + \frac{d}{dx} \int_1^{3x+1} \sin(t^4) \, dt = -\sin((2x)^4) \cdot 2 + \sin((3x + 1)^4) \cdot 3 \]
(j) \[ \int \frac{x^2}{\sqrt{1-x}} \, dx \] Here, we'll try \( u = 1-x \), so that \( du = -dx \). But we still have \( x^2 \) in the numerator. Use \( u = 1-x \) so that \( x = 1-u \) and \( x^2 = (1-u)^2 = 1-2u+u^2 \), and now do the full substitution:

\[
\int \frac{x^2}{\sqrt{1-x}} \, dx = - \int \frac{1-2u+u^2}{\sqrt{u}} \, du = - \int (u^{-1/2} - 2u^{1/2} + u^{3/2}) \, du = - \left( 2u^{1/2} - \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} \right) + C = -2\sqrt{1-x} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C
\]

Note: Since \( C \) is arbitrary, you could have either \( C \) or \(-C\) in your answer.

4. If \( f \) is continuous and \( \int_0^4 f(x) \, dx = 10 \), find \( \int_0^2 f(2x) \, dx \). With that \( 2x \), we should be tempted to use \( u = 2x \), \( du = 2 \, dx \) or \( \frac{1}{2} \, du = dx \). Now, for the bounds, if \( x = 0 \), then \( u = 0 \) and if \( x = 2 \), \( u = 4 \). This says:

\[
\int_0^2 f(2x) \, dx = \frac{1}{2} \int_0^4 f(u) \, du = \frac{1}{2} \cdot 10 = 5
\]

5. If \( g(x) = \int_0^x \frac{1}{1+u^2} \, dt \), find where \( g \) is concave up. To find where \( g \) is concave up, we must compute its second derivative and find where it is positive:

\[
g'(x) = \frac{1}{1+x+x^2} = (1+x+x^2)^{-1} \quad g''(x) = \frac{-(1+2x)}{(1+x+x^2)^2}
\]

Since the denominator is always positive, we need look only for where the numerator is positive:

\[-1 - 2x > 0 \Rightarrow -2x > 1 \Rightarrow x < \frac{1}{2}\]

If \( x < \frac{1}{2} \), then \( g''(x) > 0 \), so \( g \) is concave up.

6. If \( \int_0^6 f(x) \, dx = 10 \) and \( \int_0^4 f(x) \, dx = 7 \), find \( \int_4^6 f(x) \, dx \). By a property of integrals,

\[
\int_4^6 f(x) \, dx = \int_0^6 f(x) \, dx - \int_0^4 f(x) \, dx = 10 - 7 = 3
\]

**Challenge Problems!**

If you breezed through the previous questions, and are looking for something more challenging, try these!

1. \[
\frac{d^2}{dx^2} \int_0^x \left( \int_1^{\sin(t)} \sqrt{1+u^4} \, du \right) \, dt
\]

First, let \( F(t) = \int_1^{\sin(t)} \sqrt{1+u^4} \, du \), so that the first derivative is:

\[
\frac{d}{dx} \int_0^x F(t) \, dt = F(x) = \int_1^{\sin(x)} \sqrt{1+u^4} \, du
\]

Now the next derivative is our standard problem:

\[
\frac{d}{dx} \int_1^{\sin(x)} \sqrt{1+u^4} \, du = \sqrt{1+(\sin(x))^4} \cdot \cos(x)
\]
2. If \( f \) is a differentiable function so that: 
\[
\int_0^x f(t) \, dt = (f(x))^2
\]
for all \( x \), find \( f \).

We can remove the integral by differentiating both sides:

\[
f(x) = 2f(x) \cdot f'(x) \Rightarrow f(x) - 2f(x) \cdot f'(x) = 0 \Rightarrow f(x) (1 - 2f'(x)) = 0
\]

This says that either \( f(x) = 0 \) or \( 1 - 2f'(x) = 0 \). For the first case, we see that \( f(x) = 0 \) will solve our original function, since \( \int_0^x 0 \, dx = 0 \) for all \( x \).

In the second case, \( f'(x) = \frac{1}{2} \), so \( f(x) = \frac{1}{2}x + C \). To get the value of \( C \), notice in the original equation that if \( x = 0 \), then:

\[
\int_0^0 f(x) \, dx = (f(0))^2 \Rightarrow f(0) = 0
\]

Thus, \( C = 0 \).

So, we have two possibilities: \( f(x) = 0 \) or \( f(x) = \frac{1}{2}x \).

3. Find

\[
\lim_{h \to 0} \frac{1}{h} \int_2^{2+h} \sqrt{1 + t^3} \, dt
\]

Do we recognize this as a derivative? If \( g(x) = \int_0^x f(t) \, dt \), then

\[
g'(2) = \lim_{h \to 0} \frac{1}{h} (g(2 + h) - g(2)) = \lim_{h \to 0} \frac{1}{h} \left( \int_0^{2+h} f(t) \, dt - \int_0^2 f(t) \, dt \right) = \lim_{h \to 0} \frac{1}{h} \int_2^{2+h} f(t) \, dt
\]

In particular, if \( g(x) = \int_0^x \sqrt{1 + t^3} \, dt \), then \( g'(x) = \sqrt{1 + t^3} \), so

\[
\lim_{h \to 0} \frac{1}{h} \int_2^{2+h} \sqrt{1 + t^3} \, dt = g'(2) = \sqrt{1 + 2^3} = \sqrt{9} = 3
\]

Note: the use of 0 in the definition of \( g \) was arbitrary- we could’ve used any constant greater than \(-1\).