Does it converge or diverge? If it converges, find its value (if possible).

1. \( \sum_{n=2}^{\infty} \frac{1}{n - \sqrt{n}} \)

   The terms of the sum go to zero. It looks similar to \( \sum \frac{1}{n} \), which diverges.

   We also note that the terms of the sum are positive. We compare them:

   \[
   \lim_{n \to \infty} \frac{n - \sqrt{n}}{1} = \lim_{n \to \infty} \frac{n}{n - \sqrt{n}} = \lim_{n \to \infty} \frac{1}{1 - \frac{1}{\sqrt{n}}} = 1
   \]

   The series diverges by the limit comparison test, with \( \sum(1/n) \).

2. \( \left\{ \frac{n}{1 + \sqrt{n}} \right\} \)

   In this case, we simply take the limit:

   \[
   \lim_{n \to \infty} \frac{n}{1 + \sqrt{n}} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n} + 1} = \infty
   \]

   The sequence diverges.

3. \( \sum_{n=2}^{\infty} \frac{n^2 + 1}{n^3 - 1} \)

   The terms of the sum go to zero, since there is an \( n^2 \) in the numerator, and \( n^3 \) in the denominator. In fact, it looks like \( \sum \frac{1}{n} \), so we compare it to that:

   \[
   \lim_{n \to \infty} \frac{n^2 - 1}{n} = \lim_{n \to \infty} \frac{n^3 - n}{n^3 - 1} = 1
   \]

   Therefore, the series diverges by the limit comparison test, with \( \sum \frac{1}{n} \).

4. \( \sum_{n=1}^{\infty} \frac{5 - 2\sqrt{n}}{n^3} \)

   We can temporarily break this apart to see if the pieces converge:

   \[
   \sum_{n=1}^{\infty} \frac{5 - 2\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{5}{n^3} - 2 \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3}
   \]

   Both of these are \( p \)-series, the first with \( p = 3 \), the second with \( p = \frac{5}{2} \), therefore they converge separately, and so the sum also converges.

5. \( \sum_{n=1}^{\infty} (-6)^{n-1} 5^{1-n} \)

   First, let’s rewrite the terms of the sum:

   \[
   (-6)^{n-1} 5^{1-n} = \frac{(-6)^{n-1}}{5^{n-1}} = \left( \frac{-6}{5} \right)^{n-1}
   \]

   1
so that this is a geometric series with \( r = \frac{-6}{5} \). Since \( |r| > 1 \), this series diverges.

6. \( \left\{ \frac{n!}{(n+2)!} \right\} \)

We first simplify:
\[
\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}
\]
so the limit as \( n \to \infty \) is 0.

7. \( \sum_{n=1}^{\infty} \ln \left( \frac{n}{n+1} \right) \)

A little tricky... First, note that this can be written as \( \ln(n) - \ln(n+1) \).
Now, let’s write out the \( n^{th} \) partial sum:
\[
S_n = \ln(1) - \ln(2) + \ln(2) - \ln(3) + \ln(3) - \ln(4) + \ldots + \ln(n) - \ln(n+1)
\]
with cancellations,
\[
S_n = 0 - \ln(n+1)
\]
Now, the limit of \( S_n \) as \( n \to \infty \) is \( -\infty \), so the sum diverges.

8. \( \sum_{n=2}^{\infty} \frac{3^n + 2^n}{6^n} \)

A sum of geometric series:
\[
\sum_{n=2}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=2}^{\infty} \left( \frac{1}{2} \right)^n + \sum_{n=2}^{\infty} \left( \frac{1}{3} \right)^n = \frac{(1/2)^2}{1 - (1/2)} + \frac{(1/3)^2}{1 - (1/3)} = \frac{2}{3}
\]

9. \( \{ \sin \left( \frac{n\pi}{2} \right) \} \)

Write out the first few terms of the sequence:
1, 0, -1, 0, 1, 0, -1, . . .
so the sequence diverges.

10. \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \)

First, we see the terms go to zero like \( \frac{1}{n^3} \).
\[
\lim_{n \to \infty} \frac{n^3}{n(n+1)(n+2)} = 1
\]
so the series converges by the limit comparison test.
11. \( \sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^{\sqrt{n}}} \)

First, do the terms go to zero? The maximum value of the sine function is 1, and all terms of the sum are positive, so:

\[
\frac{\sin^2(n)}{n^{\sqrt{n}}} \leq \frac{1}{n^{\sqrt{n}}}
\]

so the terms do go to zero. Actually, we’ve also done a direct comparison with the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \), which converges.

12. \( \sum_{n=1}^{\infty} \frac{n}{(n + 1)2^n} \)

It looks like the terms are going to zero like \( \frac{1}{2^n} \), so let’s compare it to \( \sum \frac{1}{2^n} \), which is a convergent geometric series.

\[
\frac{n}{n + 1} \cdot \frac{1}{2^n} \leq \frac{1}{2^n}
\]

So the series converges by a direct comparison.

Evaluate, if possible.

1. \( \int \frac{x^3}{x^3 + 1} \, dx \)

Do long division first!

\[
x^3 + 1 = 1 - \frac{1}{x^3 + 1}
\]

Can we factor \( x^3 + 1 \)? We see \( x = -1 \) gives 0, so \( x + 1 \) can be factored out. Using long division,

\[
\frac{x^3 + 1}{x + 1} = x^2 - x + 1
\]

so that \( x^3 + 1 = (x + 1)(x^2 - x + 1) \) (NOTE: On the exam, you will be able to factor the polynomial easier than this!)

By Partial Fractions,

\[
\frac{x^3}{x^3 + 1} = 1 - \frac{1}{(x + 1)(x^2 - x + 1)} = 1 + \frac{1}{3} \cdot \frac{1}{x + 1} + \frac{1}{3} \cdot \frac{2 - x}{x^2 - x + 1}
\]

Now you have to complete the square to finish things off, and after some long algebra,

\[
x + \frac{1}{6} \ln(x^2 - x + 1) - \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2x - 1}{\sqrt{3}} \right) - \frac{1}{3} \ln(x + 1)
\]

NOTE: This was a complicated exercise! If you made it through this one, you could probably stop now- You’re ready! There won’t be anything this complex on the exam...
2. $\int_0^1 \frac{1}{2 - 3x} \, dx$

There is a vertical asymptote at $x = \frac{2}{3}$, so we need to split the integral there:

$$\int_0^1 \frac{1}{2 - 3x} \, dx = \lim_{T \to 2/3} \int_0^T \frac{1}{2 - 3x} \, dx + \lim_{T \to 2/3} \frac{1}{2 - 3x} \, dx$$

Integrate by taking $u = 2 - 3x$, and we get that the antiderivative is $-\frac{1}{3} \ln |2 - 3x|$. Now, take the limits- we’ll do one here:

$$\lim_{T \to 2/3^-} -\frac{1}{3} \ln \left( \frac{1}{2 - 3T} \right) + \frac{1}{3} \cdot \frac{1}{2} = \lim_{T \to 2/3^-} +\frac{1}{3} \ln (2 - 3T) + \frac{1}{3} \cdot \frac{1}{2}$$

which diverges, since 0 is a vertical asymptote for $\ln(x)$.

3. $\int \frac{1}{x^4 - x^2} \, dx$

Factor and use partial fractions:

$$\int \frac{1}{x^2(x - 1)} \, dx = \frac{1}{x} + \frac{1}{2} \ln(x - 1) - \frac{1}{2} \ln(x + 1)$$

4. $\int_1^\infty \frac{1}{1 + e^x} \, dx$

Use $u = 1 + e^x$, so $du = e^x \, dx$, so that $du = (u - 1)dx$.

Substitution gives:

$$\int \frac{1}{u(u - 1)} \, du = \int \frac{-1}{u} + \frac{1}{u - 1} \, du$$

Antidifferentiate, and we get:

$$-\ln (1 + e^x) + \ln(e^x) = \ln \left( \frac{e^x}{1 + e^x} \right)$$

Take the appropriate limit to get an answer of $\ln(2)$.

5. $\int_0^\infty \frac{dx}{(x + 1)^2(x + 2)} \, dx$

Use partial fractions:

$$\int \frac{dx}{(x + 1)^2(x + 2)} \, dx = \int \frac{1}{(x + 1)^2} + \frac{1}{x + 2} - \frac{1}{x - 1} \, dx$$

Antidifferentiate to get:

$$-\frac{1}{x + 1} + \ln |x + 2| - \ln |x + 1| = -\frac{1}{x + 1} + \ln \left| \frac{x + 2}{x + 1} \right|$$

And, take the limit to get $-1 + \ln(2)$.
6. \( \int \frac{5x^2 + 3x - 2}{x^3 + 2x^2} \, dx \)

Use partial fractions to get:

\[ \int \frac{-1}{x^2} + \frac{2}{x} + \frac{3}{x+2} \, dx \]

And integrate to get:

\[ \frac{1}{x} + 2 \ln(x) + 3 \ln(x + 2) \]