Homework Hints for Integration by Parts (7.1)

3.

sign	u	dv
$+$	x	$\cos(5x)$
$-$	1	$(1/5)\sin(5x)$
$+$	0	$-(1/25)\cos(5x)$

$$\int x\cos(5x)\,dx = \frac{1}{5}x\sin(5x) + \frac{1}{25}\cos(5x) + C$$

7.

sign	u	dv
$+$	x^2+2x	$\cos(x)$
$-$	$2x+2$	$\sin(x)$
$+$	2	$-\cos(x)$
$-$	0	$-\sin(x)$

$$\int (x^2+2x)\cos(x)\,dx = (x^2+2x)\sin(x) + 2(x+1)\cos(x) - 2\sin(x) + C$$

9. Easiest to use a property of logarithms first: $\ln(\sqrt[3]{x}) = \ln(x^{1/3}) = \frac{1}{3}\ln(x)$, so that the integral becomes:

$$\int \ln\sqrt[3]{x}\,dx = \frac{1}{3}\int \ln(x)\,dx$$

Now use integration by parts with $u = \ln(x)$ and $dv = 1\,dx$.

15. It's easy to differentiate $(\ln(x))^2$, so we'll go with that:

sign	u	dv
$+$	$(\ln(x))^2$	1
$-$	$2\ln(x)/x$	x

$$\int (\ln(x))^2\,dx = x(\ln(x))^2 - 2\int \ln(x)\,dx$$

and we find $\int \ln(x)\,dx$ by using IBP again.

17. (**Also see Example 4 in the text!**) We integrate by parts twice in order to get the same integral to appear on both sides of the equation.

sign	u	dv
$+$	$\sin(3\theta)$	$e^{2\theta}$
$-$	$3\cos(3\theta)$	$(1/2)e^{2\theta}$
$+$	$-9\sin(3\theta)$	$(1/4)e^{2\theta}$

$$\int e^{2\theta}\sin(3\theta)\,d\theta = \frac{1}{2}e^{2\theta}\sin(3\theta) - \frac{3}{4}e^{2\theta}\cos(3\theta) - \frac{9}{4}\int e^{2\theta}\sin(3\theta)\,d\theta$$

$$\frac{13}{4}\int e^{2\theta}\sin(3\theta)\,d\theta = \frac{1}{2}e^{2\theta}\sin(3\theta) - \frac{3}{4}e^{2\theta}\cos(3\theta)$$

$$\int e^{2\theta}\sin(3\theta)\,d\theta = \frac{2}{13}e^{2\theta}\sin(3\theta) - \frac{3}{13}e^{2\theta}\cos(3\theta) + C$$

27. Our usual technique of differentiating until we get zero may not work here, since we'll need to antidifferentiate $\ln(x)$ that many times as well. Let's put $\ln(r)$ in the middle column instead (since the derivative is $1/r$), and this is our first definite integral:

sign	u	dv
$+$	$\ln(r)$	r^3
$-$	$1/r$	$(1/4)r^4$

$$\int_1^3 r^3 \ln(r)\,dr = \left(\frac{1}{4}r^4 \ln(r)\right|_1^3 - \frac{1}{4}\int_1^3 r^3\,dr = \left(\frac{1}{4}r^4 \ln(r) - \frac{1}{16}r^4\right|_1^3 =$$
$$\frac{4 \cdot 3^4 \ln(3) - 3^4 + 1}{16} = \frac{81}{4}\ln(3) - 5$$

29. Put y in the middle column and differentiate it (the exponential is straightforward to antidifferentiate):

sign	u	dv
$+$	y	e^{-2y}
$-$	1	$-(1/2)\mathrm{e}^{-2y}$
$+$	0	$(1/4)\mathrm{e}^{-2y}$

$$\int_0^1 \frac{y}{\mathrm{e}^{2y}}\,dy = \left(-\frac{1}{2}y\mathrm{e}^{-2y} - \frac{1}{4}\mathrm{e}^{-2y}\right|_0^1 = -\frac{3}{4}\mathrm{e}^{-2} + \frac{1}{4}$$

30. If we put $\tan^{-1}(1/x)$ into the middle column of the integration by parts table, we need its derivative:

$$\frac{d}{dx}(\tan^{-1}(1/x)) = \frac{1}{1+(1/x)^2} \cdot -\frac{1}{x^2} = \frac{x^2}{x^2+1} \cdot -\frac{1}{x^2} = -\frac{1}{x^2+1}$$

Using that, we have the following (note also that this is a definite integral):

sign	u	dv
$+$	$\tan^{-1}(1/x)$	1
$-$	$-1/(x^2+1)$	x

$$x\tan^{-1}(1/x)\Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{x^2+1}\,dx$$

Here, using a 30-60-90 triangle and a 45-45-90 triangle, we have:

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \qquad \text{and} \qquad \tan^{-1}(1) = \frac{\pi}{4}$$

After evaluating the previous expression, and integrating using u, du substitution, we get:

$$\frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\ln(2)$$

31. I had said previously that you wouldn't need to know the derivative of the inverse cosine, so you may skip this problem. In its place, suppose we do $\int_0^{1/2} \sin^{-1}(x)\,dx$ (the two problems are very similar). Then setting up integration by parts is the usual:

sign	u	dv
$+$	$\sin^{-1}(x)$	1
$-$	$1/\sqrt{1-x^2}$	x

$$x\sin^{-1}(x)\Big|_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$$

Like the problem with the inverse tangent, we should be able to compute $\sin^{-1}(1/2)$ using a triangle- In this case, we get $\pi/6$. For the integral, let $w = 1 - x^2$ so $dw = -2x\,dx$, and the integral becomes

$$\frac{1}{2}\int w^{-1/2}dw = w^{1/2} = \sqrt{1-x^2}$$

Putting this altogether, we get:

$$\left(\frac{\pi}{12} - 0\right) + \left(\frac{\sqrt{3}}{2} - 1\right)$$

32. This one's a little long, but each piece is do-able. We'll end up performing integration by parts twice.

Let $u = (\ln(x))^2$ and $dv = x^{-3}\,dx$. Then

sign	u	dv
$+$	$(\ln(x))^2$	x^{-3}
$-$	$2\ln(x)/x$	$-(1/2)x^{-2}$

Now we have:

$$\int \frac{(\ln(x))^2}{x^3}\,dx = -\frac{1}{2}\frac{(\ln(x))^2}{x^2} + \int \frac{\ln(x)}{x^3}\,dx$$

And, the second integral above can be evaluated using integration by parts again:

sign	u	dv
$+$	$\ln(x)$	x^{-3}
$-$	$1/x$	$-(1/2)x^{-2}$

$$\int \frac{\ln(x)}{x^3}\,dx = -\frac{1}{2}\frac{\ln(x)}{x^2} + \frac{1}{2}\int x^{-3}\,dx = -\frac{1}{2}\frac{\ln(x)}{x^2} - \frac{1}{4x^2}$$

Put it all together:

$$\int_1^2 \frac{(\ln(x))^2}{x^3}\,dx = \left(-\frac{1}{2}\frac{(\ln(x))^2}{x^2} - \frac{1}{2}\frac{\ln(x)}{x^2} - \frac{1}{4x^2}\right|_1^2 = -\frac{1}{8}(\ln(2))^2 - \frac{1}{8}\ln(2) + \frac{3}{16}$$

37. This is a handy example to keep in mind! For u, du substitution, we'll use w since we're using u, v in the integration by parts.

If $w = \sqrt{x}$, then $w^2 = x$, and $2w\,dw = dx$. Substituting this into the integral,

$$\int \cos(\sqrt{x})\,dx = \int \cos(w)(2w\,dw) = 2\int w\,\cos(w)\,dw$$

Now we can integrate by parts as usual:

sign	u	dv
$+$	w	$\cos(w)$
$-$	1	$\sin(w)$
$+$	0	$-\cos(w)$

$$2\int w\cos(w)\,dw = 2w\sin(w) + 2\cos(w) + C$$

or $2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C$

41. Similar to 37. Let $w = x + 1$ so that $x = w - 1$, and we get

$$\int x\ln(x+1)\,dx = \int (w-1)\ln(w)\,dw$$

Use integration by parts like last time:

sign	u	dv
$+$	$\ln(w)$	$w-1$
$-$	$1/w$	$(1/2)w^2 - w$

$$\int (w-1)\ln(w)\,dw = \frac{w^2-2w}{2}\ln(w) - \int \frac{1}{2}w - 1\,dw$$

and so on.

47. We'll wait on this one...

63. When setting up the radius that x is a negative number, so the radius will be "right-left", or $1 - x$.

Following through with the shell, we have:

$$\int_{-1}^{0} 2\pi(1-x)\mathrm{e}^{-x}\,dx$$

which we integrate using parts.

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68. This kind of problem is where the "integration by parts using a table" makes it very easy. With the given integral, we'll put $f(x)$ in the middle column, and $g''(x)$ in the last column. Then:

$$\int_a^b f(x)g''(x)\,dx \quad \Rightarrow$$

sign	u	dv
$+$	$f(x)$	$g''(x)$
$-$	$f'(x)$	$g'(x)$
$+$	$f''(x)$	$g(x)$

Therefore,

$$\int_0^a f(x)g''(x)\,dx = (f(x)g'(x) - f'(x)g(x)|_0^a + \int_a^a f''(x)\,g(x)\,dx$$