Review Solutions

1. What does it mean to say that a series "converges" (I'm looking for the definition; be sure you define any notation you use).

   SOLUTION: Suppose we are given the series $\sum_{k=1}^{\infty} a_k$, and define the n^{th} partial sum:

   $$S_n = \sum_{k=1}^{n} a_k.$$

   Then we say the series converges if $\lim_{n\to\infty} S_n$ exists. Further, in that case, we define

   $$S = \sum_{k=1}^{\infty} a_k$$

2. If a series converges by the Integral Test, how do you estimate it's sum (or more precisely, how do you estimate the remainder, R_n)?

   SOLUTION: Given $\sum_{n=1}^{\infty} a_n$, with $f(n) = a_n$, where f is positive, decreasing and continuous, then the remainder is estimated:

   $$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

3. If a series converges by the Alternating Series Test, how do you estimate its sum (or more precisely, how do you estimate the remainder, R_n)?

   SOLUTION: Given $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, with $b_n \geq 0$, then if the Alternating Series Test applies, then b_n must be decreasing, and the limit is 0. In that case, we can estimate the remainder

   $$R_n \leq b_{n+1}$$

4. If a series has radius of convergence ρ, can you predict the radius of convergence of the derivative of the series? For the antiderivative?

   SOLUTION: Yes- The radius of convergence doesn't change (although the endpoints of the interval of convergence may change).

5. Does the given sequence or series converge or diverge? If the series converges, is it absolute or conditional?

   (a) $\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$

SOLUTION: Using the dominating terms, this looks a lot like $\sum \frac{1}{n}$, so we use the limit comparison (note that both series, the one given and the template, have all positive terms)

$$\lim_{n\to\infty} \frac{\frac{1}{n-\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n-\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{\sqrt{n}}} = 1$$

Therefore, both series diverge together (by the limit comparison test).

(b) $\left\{\frac{n}{1+\sqrt{n}}\right\}$

SOLUTION: Take the limit; You can use L'Hospital's rule if you like. To be precise, we ought to change notation to x (since you cannot formally take the derivative of a discrete sequence):

$$\lim_{x\to\infty} \frac{x}{1+\sqrt{x}} = \lim_{x\to\infty} \frac{1}{1/2\sqrt{x}} = \lim_{x\to\infty} 2\sqrt{x} = \infty$$

Therefore, the sequence diverges.

(c) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2+1}$

SOLUTION: This one is very much like $\sum(-1)^n/n$ (the alternating harmonic series, which converges only conditionally). We will then show the series does NOT converge absolutely, then we use the Alternating Series Test to show the series converges conditionally:

- Absolute Convergence: The series $\sum n/(n^2+1)$ diverges like $\sum 1/n$, which we show using the limit comparison test:

$$\lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

  And, since $\sum 1/n$ diverges, then so does $\sum n/(n^2+1)$ by the limit comparison test.

- Conditional Convergence: The series alternates in sign, and let $b_n = n/(n^2+1)$. To show b_n decreases, we check that the derivative is negative:

$$f(x) = \frac{x}{x^2+1} \quad \Rightarrow \quad f'(x) = \frac{(x^2+1) - x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

  and the numerator is clearly negative for $x > 1$ (so b_n is decreasing). Further,

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{2n} = 0$$

  Therefore, by the Alternating Series Test, the series converges conditionally.

(d) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

SOLUTION: Use the Test for Divergence. The terms go to $\ln(1/3) \neq 0$, so the series diverges.

(e) $\sum_{n=1}^{\infty} (-6)^{n-1}5^{1-n}$

SOLUTION: This one looks like it might be a geometric series, so we put it in that form so we can see if it converges, then find the sum:

$$\sum_{n=1}^{\infty} (-6)^{n-1}5^{1-n} = \sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{-6}{5}\right)^{n-1}$$

Since $|r| = 6/5 > 1$, the series diverges.

SOLUTION: We first simplify:

$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}$$

so the limit as $n \to \infty$ is 0.

(f) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{5^n n!}$

SOLUTION: With factorials and powers, use the Ratio Test. Because all terms are always positive, we can drop the absolute value signs (if it converges, it would be absolute convergence). Before taking the limit, we can simplify algebraically:

$$\frac{a_{n+1}}{a_n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{5^{n+1}(n+1)!} \cdot \frac{5^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2n+1}{5(n+1)} = \frac{2n+1}{5n+5}$$

Now, take the limit:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{5+\frac{5}{n}} = \frac{2}{5}$$

Since the limit is less than 1, the series converges (absolutely) by the Ratio Test.

(g) $\sum_{n=2}^{\infty} \frac{3^n + 2^n}{6^n}$

A sum of (convergent) geometric series is also convergent. In fact, we can find the sum to which the series will converge:

$$\sum_{n=2}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n = \frac{(1/2)^2}{1-(1/2)} + \frac{(1/3)^2}{1-(1/3)} = \frac{2}{3}$$

(h) $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$

SOLUTION: Write out the first few terms of the sequence:

$$1, 0, -1, 0, 1, 0, -1, \ldots$$

so the sequence diverges.

(i) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

SOLUTION: We see the terms go to zero like $\frac{1}{n^3}$ (that would be a convergent p series). Therefore, use the limit comparison test:

$$\lim_{n\to\infty} \frac{n^3}{n(n+1)(n+2)} = 1$$

so the series converges by the limit comparison test.

NOTE: Did you try to use the Ratio Test? The Ratio (and Root) tests always give an inconclusive answer for any $p-$series.

(j) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n\sqrt{n}}$

SOLUTION: First, do the terms go to zero? The maximum value of the sine function is 1, and all terms of the sum are positive, so:

$$\frac{\sin^2(n)}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

so the terms do go to zero. Actually, we've also done a direct comparison with the $p-$series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges.

(k) $\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$

SOLUTION: Ratio Test (note that the negative sign is meaningless since $(-1)^{2n} = 1$). Start with some algebra to simplify before taking the limit:

$$\frac{5^{2n+2}}{(n+1)^2 9^{n+1}} \cdot \frac{n^2 9^n}{5^{2n}} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{5^{2n}5^2}{5^{2n}} \cdot \frac{9^n}{9^{n+1}} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{25}{9}$$

The limit as $n \to \infty$ is 25/9:

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \cdot \frac{25}{9} = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^2 \cdot \frac{25}{9} = \frac{25}{9} > 1$$

Therefore, the series diverges by the Ratio Test.

(l) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

SOLUTION: If you attempt to use the Ratio Test, you would get an inconclusive result. You might recognize that this problem is set up for the Integral Test with

$$f(x) = \frac{1}{x\sqrt{\ln(x)}}$$

so that f is positive, decreasing and continuous for $x > 1$. When integrating, we can let $\ln(x) = u$ and do a u, du substitution. Notice that if $x = 1$, then $u = 0$, and as $x \to \infty$, then $u \to \infty$ as well:

$$\int_1^\infty \frac{1}{x\sqrt{\ln(x)}}\,dx = \int_0^\infty u^{-1/2}\,du = \left(2u^{1/2}\right|_0^\infty \to \infty$$

The integral diverges, so the sum diverges as well.

6. Evaluate the integral or show it diverges:

(a) $\displaystyle\int_0^1 \frac{x-1}{\sqrt{x}}\,dx = \lim_{T\to 0^+}\int_T^1 \sqrt{x} - \frac{1}{\sqrt{x}}\,dx = \lim_{T\to 0^+}\left(\frac{2}{3}x^{3/2} - 2x^{1/2}\right|_T^1 = \frac{2}{3} - 2 = -\frac{4}{3}$
Therefore, the integral converges (to $-4/3$).

(b) $\displaystyle\int_2^\infty \frac{1}{x\ln(x)}\,dx = \int_{\ln(2)}^\infty \frac{1}{u}\,du$, where $u = \ln(x)$. Integrating, we get:

$$\ln(u)|_{\ln(2)}^\infty \to \infty$$

So the integral diverges.

(c) $\displaystyle\int_0^\infty x^3 \mathrm{e}^{-x^4}\,dx$
Let $u = x^4$, then perform u, du substitution as before:

$$\frac{1}{4}\int_0^\infty \mathrm{e}^{-u}\,du = -\frac{1}{4}\mathrm{e}^{-u}\Big|_0^\infty = -\frac{1}{4}\left(0 - \mathrm{e}^0\right) = \frac{1}{4}.$$

The integral converges (to $1/4$).

7. Show that the integral $\displaystyle\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$ converges or diverges. HINT: Do not try to compute the antiderivative. Be clear as to your justification.

SOLUTION: Since $\sin^2(x) \le 1$ for all x, then

$$\frac{\sin^2(x)}{x^2} \le \frac{1}{x^2} \qquad \text{for all } x$$

and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, then by the comparison test (for integrals), the original integral converges.

8. Find the sum of the series

NOTE: We only know two ways of finding the sum for a convergent series- Either by using the Geometric Series or by using the Taylor Series of a template series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{2n}}$

SOLUTION: Do some algebra first. This should look like a geometric series(?)

$$\frac{(-3)^{n-1}}{2^{2n}} = \frac{(-3)^n(-3)^{-1}}{(2^2)^n} = -\frac{1}{3} \cdot \left(-\frac{3}{4}\right)^n$$

Now, this is a convergent series with $a = -1/3$ and $r = -3/4$. The sum is:

$$\frac{(-1/3)(-3/4)}{1+\frac{3}{4}} = \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{7}$$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(x-3)^{2n}}{3^n}$

This is a geometric series with $r = \frac{(x-3)^2}{3}$. Putting it into the formula for the sum,

$$\frac{\left(\frac{(x-3)^2}{3}\right)^2}{1-\frac{(x-3)^2}{3}} = \frac{(x-3)^4}{9} \cdot \frac{3}{3-(x-3)^2} = \frac{3(x-3)^4}{3-(x-3)^2}$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n!)}$

SOLUTION: The series is a cosine series (see the even powers of x?). We might do a little algebra first:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n!)} \qquad \Rightarrow \qquad \sum_{n=0}^{\infty} \frac{(-1)^n(\pi/3)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

9. (a) $\displaystyle\sum \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

SOLUTION:

Use the Ratio Test (and remember to use the absolute value signs!). First a little algebra:

$$\frac{(n+1)!|x|^{n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!|x|^n} = \frac{n+1}{2n+1}|x|$$

Now take the limit and apply the Ratio test:

$$|x| \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{|x|}{2} < 1 \quad \Rightarrow \quad |x| < 2$$

Therefore, the radius of convergence is 2.

(b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n^2 5^n}$

SOLUTION: Use the Ratio test- First simplify.

$$\frac{|x|^{n+1}}{(n+1)^2 5^{n+1}} \cdot \frac{n^2 5^n}{|x|^n} = \left(\frac{n}{n+1}\right)^2 \cdot \frac{|x|}{5}$$

Now take the limit and apply the test:

$$\frac{|x|}{5} \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = \frac{|x|}{5}\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^2 = \frac{|x|}{5} < 1 \quad \Rightarrow \quad |x| < 5$$

The radius of convergence is 5. When we test $x = -5$ and $x = 5$, we get convergent p series ($\sum 1/n^2$ and $\sum(-1)^n/n^2$, respectively. Therefore, the interval of convergence is

$$[-5, 5]$$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n(x-3)}{\sqrt{n+3}}$

SOLUTION: Another Ratio Test... In this case, the series is centered at $x = 3$, so we'll have an exciting change of pace in calculating the interval of convergence! Here we go- As usual, do the algebra first:

$$\frac{2^{n+1}|x-3|^{n+1}}{\sqrt{n+4}} \frac{\sqrt{n+3}}{2^n|x-3|^n} = 2|x-3|\sqrt{\frac{n+3}{n+4}}$$

The limit can be brought under the radical sign since the square root is a continuous function:

$$2|x-3| \lim_{n\to\infty}\sqrt{\frac{n+3}{n+4}} = 2|x-3|\sqrt{\lim_{n\to\infty}\frac{n+3}{n+4}} = 2|x-3|$$

To apply the Ratio test, if $2|x-3| < 1$, the series will converge absolutely. Therefore, the radius of convergence is $1/2$ and to find the interval of convergence, we test the endpoints:

$$-\frac{1}{2} < x - 3 < \frac{1}{2} \quad \Rightarrow \quad \frac{5}{2} < x < \frac{7}{2}$$

If we put in $x = 5/2$, the series becomes

$$\sum \frac{2^n \cdot \left(\frac{-1}{2}\right)^n}{\sqrt{n+3}} = \sum \frac{(-1)^n}{\sqrt{n+3}}$$

This will converge by the Alternating Series Test (diverges absolutely since it is similar to a divergent p-series): (i) It is alternating. (ii) It is decreasing: $\sqrt{n+4} > \sqrt{n+3}$, so $1/\sqrt{n+4} < 1/\sqrt{n+3}$. (iii) The terms go to zero.

If we put in $x = 7/2$, we get something similar to a divergent p series, which diverges:

$$\sum \frac{1}{\sqrt{n+3}}$$

We could show it by the limit comparison test with $1/\sqrt{n}$.

Summary: The interval is $[5/2, 7/2)$

10. Use a series to evaluate the following limit: $\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$

SOLUTION: Use our template series for the sine function- In fact, write it out:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

so

$$\sin(x) - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and

$$\frac{\sin(x) - x}{x^3} = -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots$$

Evaluate this series at $x = 0$ (because we want the limit as $x \to 0$) to get the answer of $-1/6$.

Optional: We can verify our answer using L'Hospital's rule applied several times:

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin(x)}{6x} = \lim_{x\to 0} \frac{-\cos(x)}{6} = -\frac{1}{6}$$

11. Use a known template to find a series for the following:

(a) $\frac{x^2}{1+x}$

SOLUTION: This looks kinda like the sum of a geo series:

$$\frac{x^2}{1+x} = x^2 \cdot \frac{1}{1+x} = x^2 \cdot \frac{1}{1-(-x)} = x^2 \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$$

(b) $\sin(x^2)$

SOLUTION: Use the series for $\sin(x)$, then replace x by x^2 (then simplify a bit):

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} =$$

(c) $x\mathrm{e}^{2x}$

SOLUTION: Start with the series for e^x, and substitute $2x$ in where we see an x. To get the series for $x\mathrm{e}^{2x}$, multiply the series by x:

$$\mathrm{e}^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \quad \Rightarrow \quad x\mathrm{e}^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$$

12. Find the Taylor series for $f(x)$ centered at the given base point:

(a) $x^4 - 3x^2 + 1$, at $x = 1$

Set up the table:

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^4 - 3x^2 + 1$	-1
1	$4x^3 - 6x$	-2
2	$12x^2 - 6$	6
3	$24x$	24
4	24	24
5	0	0

$$\Rightarrow \quad f(x) = -1 - 2(x-1) + \frac{6}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

The other terms of the sum are zero.

Optional note: If you expand your Taylor series and simplify, you should get the original polynomial.

(b) $1/\sqrt{x}$ at $x = 9$ (just get the first four non-zero terms of the power series).

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{-1/2}$	$1/3$
1	$(-1/2)x^{-3/2}$	$-1/54$
2	$(3/4)x^{-5/2}$	$1/324$
3	$(-15/8)x^{-7/2}$	$-5/5832$

$$\Rightarrow \quad \frac{1}{\sqrt{x}} \approx \frac{1}{3} - \frac{1}{54}(x-9) + \frac{1}{648}(x-9)^2 - \frac{5}{34992}(x-9)^3$$

Sorry about those numbers- They'll be nicer for the exam.

(c) $1/x^2$ at $x = 1$. In this case, find a pattern for the n^{th} coefficient so that you can write the general series. Using this answer, find the radius of convergence.

SOLUTION: Build a table

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$(-2)x^{-3}$	-2
2	$(3)(2)x^{-4}$	$3!$
3	$-(4)(3)(2)x^{-5}$	$-4!$
$\vdots$	$\vdots$	$\vdots$
n		$(-1)^n(n+1)!$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{n!}(x-1)^n \\ &= \sum_{n=0}^{\infty} (-1)(n+1)(x-1)^n \end{aligned}$$

With the Ratio Test we get (after the algebra):

$$|x-1| \lim_{n\to\infty} \frac{n+2}{n+1} = |x-1| < 1$$

Therefore, the radius of convergence is 1. *NOTE:* We could have anticipated that, since $1/x^2$ has a vertical asymptote at $x = 0$, and our base point is $x = 1$.

13. True or False, and give a short reason:

(a) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum a_n$ is convergent.
SOLUTION: FALSE. For example, $1/n$ goes to zero, but the series diverges.

(b) If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
SOLUTION: TRUE. This is equivalent to the Test for Divergence, which says that if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

(c) The Ratio Test can be used to determine if a $p-$series is convergent.
SOLUTION: FALSE. Using the Ratio Test on a $p-$series will give a limit of 1. For example, given $\sum 1/n^p$, then

$$\lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1$$

(d) If $0 \le a_n \le b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
SOLUTION: FALSE. If $\sum a_n$ were divergent, we could then conclude that $\sum b_n$ diverges.

(e) If $a_n > 0$ for all n and $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges.
SOLUTION: TRUE. You could re-phrase the question as: If a series converges absolutely, would the corresponding alternating series converge? Yes (absolutely!).

14. It is well known that there is no "simple" antiderivative for e^{-x^2}. Find a series representation for $\int \mathrm{e}^{-x^2}\,dx$ and give the radius of convergence. HINT: Start with a template series that we know.

SOLUTION: Starting with the series for e^x, we have

$$\mathrm{e}^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, x^{2n}$$

Integrating both sides, we get:

$$\int \mathrm{e}^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1} + C = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\cdot n!} + C$$

The radius of convergence doesn't change via integration, and the original radius was ∞, so it remains ∞.

15. Suppose that $\sum_{n=0}^{\infty} c_n (x-1)^n$ converges when $x = 3$ and diverges when $x = -2$.

SOLUTION NOTE before continuing. From what is given, we know that the radius of convergence must be at least the distance from $x = 1$ (the center) to $x = 3$ (where we know we have convergence), so $\rho \ge 2$. It is possible that ρ is as large as the distance between $x = 1$ and $x = -2$ (means $2 \le \rho \le 3$), but we cannot guarantee that. What we can say is that if $|x-1| > 3$, the series must diverge.

(a) What is the largest interval for x on which we can guarantee that the series converges.

SOLUTION: From the note, we MUST have convergence at least on the interval: $(-1, 3]$, but that is all we can guarantee.

(b) What can be said about the sum: $\sum_{n=0}^{\infty}(-1)^n c_n$

SOLUTION: This is asking if we have convergence if $x - 1 = -1$, or at $x = 0$. That point is in the interval given in part (a), so the sum must converge.

(c) What can be said about the sum: $\sum_{n=0}^{\infty} c_n 4^n$

SOLUTION: Here we have $|x - 1| = 4$ (so $x = 5$, but that's irrelevant). From the note, this is on the interval where we must have divergence.

16. Find the Maclaurin series for $\ln(x + 1)$ and find the radius of convergence. You may do it from scratch or by using a template series.

SOLUTION (using a template series): We notice that

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty}(-x)^n\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C = C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

Since $\ln(1+0) = \ln(1) = 0$, we see that $C = 0$, and we could change the starting index (not necessary):

$$\ln(1 + x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

SOLUTION (from scratch):

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1 + x)$	0
1	$(1 + x)^{-1}$	1
2	$-(1 + x)^{-2}$	-1
3	$2(1 + x)^{-3}$	2
4	$-3 \cdot 2(1 + x)^{-4}$	$-3!$
5	$4 \cdot 3 \cdot 2(1 + x)^{-5}$	$4!$
$\vdots$		
n		$(-1)^{n+1}(n - 1)!$

$$\Rightarrow \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n-1)!}{n!}x^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

Notice that the pattern in the table is true only for $n = 1, 2, 3, \cdots$, and so we started the sum with $n = 1$.

17. Find the sum: $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

SOLUTION: After some analysis, we see that this could be written as:

$$\frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{3^n}{n!}$$

Compare that to the series for e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

Therefore, evaluating everything at $x = 3$, we have:

$$e^3 = 1 + \sum_{n=1}^{\infty} \frac{3^n}{n!}$$

so we can solve for our sum:

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} = e^3 - 1$$

18. Use the remainder for the Taylor series to approximate how large the error will be if I use a 3rd order ($n = 3$) Maclaurin series to estimate $\sin(x)$ at $x = 1/2$.

SOLUTION: There are two ways to approach the remainder- Either by using the estimate we obtained in 11.10 (for R_n), or by seeing that the series we have is an alternating series. That is:

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

so that

$$\sin(1/2) = \frac{1}{2} - \frac{1}{3! \cdot 2^3} + \frac{1}{5! \cdot 2^5} - \frac{1}{7! \cdot 2^7} + \cdots$$

If we see this as an alternating series, then

$$R_3 \le b_4 \quad \Rightarrow \quad R_3 \le \frac{1}{7! \cdot 2^7} \approx 1.55 \times 10^{-6}$$

Using the Remainder Theorem for a Taylor series, if $|f^{(n+1)}(x)| \le M$, then

$$R_n \le \frac{M}{(n+1)!}|x - a|^{n+1}$$

In this case, the remainder estimate is quite a bit larger ($M = 1$, $a = 0$, and $x = 1/2$):

$$R_3 \le \frac{1}{4! \cdot 2^4} \approx 2.6 \times 10^{-3}$$

Side Note: The exact value of the remainder is approximately 1.54×10^{-6}, so the remainder using the Alternating Series was a lot better in this case!

19. Let $a_n = \frac{2n}{3n+1}$

(a) Determine whether $\{a_n\}$ is convergent.
SOLUTION: The *sequence* a_n converges to:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$$

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.
SOLUTION: The *series* diverges by the Test for Divergence. In part (a), we showed that $a_n \to 2/3 \neq 0$, so the series $\sum_{n=1}^{\infty} a_n$ diverges.

20. Put the following quantities in order, from smallest to largest, if $f(x)$ is a positive, continuous, decreasing function, $a_n = f(n)$, and R_n is the remainder after using n terms of the sum:

(SOLUTION given below, see the sketches on p. 718.):

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

21. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

(a) Show that the series converges absolutely by using the Integral Test (if appropriate- Check it).
SOLUTION: Here, $f(x) = \frac{1}{x^4}$, so f is positive, decreasing (for $x > 0$) and continuous (for $x > 0$), so the integral test is appropriate. Integrating,

$$\int_1^{\infty} x^{-4}\,dx = \lim_{T\to\infty} \left(-\frac{1}{3} x^{-3} \Big|_1^T \right) = -\frac{1}{3} \left(\lim_{T\to\infty} \frac{1}{T^3} - 1 \right) = \frac{1}{3}$$

(b) Give an estimate of the error using the integral if we use 4 terms to estimate the sum.
SOLUTION: By "error", I meant "remainder"- Sorry about that... Using 4 terms,

$$\int_5^{\infty} \frac{1}{x^4}\,dx \leq R_4 \leq \int_4^{\infty} \frac{1}{x^4}\,dx \quad \Rightarrow \quad \frac{1}{3\cdot 5^3} \leq R_4 \leq \frac{1}{3\cdot 4^3}$$

Side note: Numerically, $0.00267 \leq R_4 \leq 0.0052$

22. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

(a) Prove the series converges by using the Alternating Series Test. Be clear about what you have to check for this!

SOLUTION: To use the Alternating Series Test, the series must be: (i) Alternating? (Check- The $(-1)^n$ term gives us that), (ii) $b_n > 0$? Check: $b_n = 1/n^4 > 0$, and (iii) The limit of b_n as $n \to \infty = 0$? Check, $\lim_n \to \infty \frac{1}{n^4} = 0$.
Therefore, the series converges. (And in fact, the series converges absolutely, but that wasn't the question).

(b) Given that the first few values are given by the following table, how many terms should we use if we want to estimate the sum correct to 3 decimal places?

	$(-1)^n/(n^4)$
$n = 1$	-1
$n = 2$	0.0625
$n = 3$	-0.01234567
$n = 4$	0.00390625
$n = 5$	-0.0016
$n = 6$	0.00077160
$n = 7$	-0.00041649
$n = 8$	0.00024414
$\vdots$	

SOLUTION: I wanted to see if we can translate the question to say that we want $b_{n+1} \leq 5 \times 10^{-4}$. If we do that, just look at the last column- We see that

$$b_7 \approx 4.16 \times 10^{-4} \leq 5 \times 10^{-4}$$

so we need 6 terms of the sum ($R_6 \leq b_7$).

23. The terms of a series are defined recursively by the equations:

$$a_1 = 2 \qquad\qquad a_{n+1} = \frac{5n+1}{4n+3} a_n$$

so, for example,

$$a_2 = \frac{6}{7} a_1 = \frac{12}{7}, \qquad a_3 = \frac{11}{11} \cdot a_2 = 1 \cdot \frac{12}{7} = \frac{12}{7}, \cdots$$

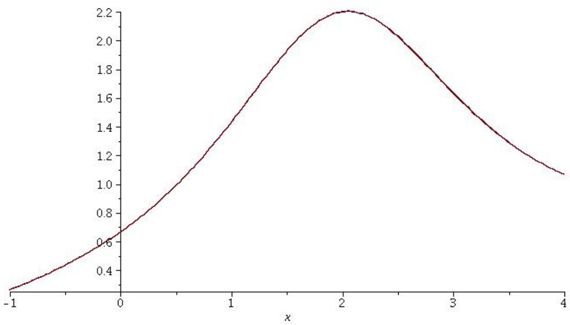
Does the series converge or diverge? (Hint: You have enough information to run a convergence test).

SOLUTION: Use the Ratio Test. I can leave off the absolute value signs, since the terms will be positive. First, let's do the algebra for the ratio:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{5n+1}{4n+3} a_n}{a_n} = \frac{5n+1}{4n+3}$$

Therefore, the limit of the ratio is $5/4$, which is larger than 1. The series diverges (by the Ratio Test).

24. Consider the graph of the function $f(x)$ below:



(a) Explain why the following is NOT the Taylor series for f centered at $x = 1$:

$$1.4 - (x-1) + 0.2(x-1)^2 - 0.2(x-1)^3 + \cdots$$

SOLUTION: From the given Taylor series, we can evaluate $f(1), f'(1), f''(1)$ (and $f'''(1)$, but we don't need that one). Here then, is it plausible that

$$f(1) = 1.4, \qquad f'(1) = -1 \qquad \frac{f''(1)}{2} = 0.2$$

The first quantity is fine. The second is wrong- The slope of the tangent line to the graph at $x = 1$ should be positive (f is increasing). Therefore, this cannot be the Taylor series for f in the graph.

(b) Explain why the following is NOT the Taylor series for f centered at $x = 2$:

$$2.2 + 0.1(x-2) + (x-2)^2 + 0.5(x-2)^4 + \cdots$$

SOLUTION: The reasoning is similar to the previous problem- Here, $f(2)$ and $f'(2)$ are plausible, but $f''(2)$ is not.