Solutions to Extra Practice in Riemann Sums

1. For each of the following integrals, write the definition using the Riemann sum (and right endpoints), but do not evaluate them:

   (a) $\displaystyle\int_2^5 \sin(3x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \sin\left(3(2+3i/n)\right)\frac{3}{n}$

   (b) $\displaystyle\int_1^3 \sqrt{1+x}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \sqrt{1+(1+2i/n)}\,\frac{2}{n}$

   (c) $\displaystyle\int_0^2 \mathrm{e}^x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \mathrm{e}^{2i/n}\frac{2}{n}$

2. For each of the following integrals, write the definition using the Riemann sum, and then evaluate them (MUST use the limit of the Riemann sum for credit, and do not re-write them using the properties of the integral):

   (a) $\displaystyle\int_2^5 x^2\,dx$

   $$\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{3i}{n}\right)^2\frac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(4+\frac{12i}{n}+\frac{9i^2}{n^2}\right)\frac{3}{n} =$$

   $$\lim_{n\to\infty}\sum_{i=1}^{n}\left(12+18\cdot\frac{n+1}{n}+\frac{27}{6}\cdot\frac{(n+1)(2n+1)}{n^2}\right) = 39$$

   (b) $\displaystyle\int_1^3 1-3x\,dx$

   $$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1-3\left(1+\frac{2i}{n}\right)\right]\frac{2}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[-2-\frac{6i}{n}\right]\frac{2}{n} = \lim_{n\to\infty}-\frac{2}{n}(2n+3(n+1)) = -10$$

   (c) $\displaystyle\int_0^5 1+2x^3\,dx$

   If we simplify the function first,

   $$f(5i/n) = 1+2(5i/n)^3 = 1+\frac{250}{n^3}i^3$$

   Using the Riemann sum, if we sum this expression for $i = 1..n$, we get

   $$\sum_{i=1}^{n} f(5i/n) = n+\frac{250}{n^3}\cdot\frac{n^2(n+1)^2}{4}$$

   Multiply by $5/n$ and take the limit, and we get:

   $$5+\frac{625}{2} = \frac{635}{2}$$

3. For each of the following Riemann sums, evaluate the limit by first recognizing it as an appropriate integral:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right) \sqrt{1 + \frac{3i}{n}}$ (Find four different integrals for this one!)

From what is given, we know that $b - a = 3$, so if a or b is given, the other can be computed. Since we see the expression

$$1 + \frac{3i}{n}$$

then we might go ahead and take $a = 1$ (so that b must be 4). In this case, the integral will be

$$\int_1^4 \sqrt{x}\, dx$$

However, a second choice would be to take $a = 0$, so that

$$f(3i/n) = \sqrt{1 + 3i/n} \quad \Rightarrow \quad f(x) = \sqrt{1 + x}$$

and the integral would be

$$\int_0^3 \sqrt{1 + x}\, dx$$

A third alternative: Let's take $a = 3$ just for fun. Then,

$$f(3 + 3i/n) = \sqrt{1 + 3i/n} = \sqrt{-2 + (3 + 3i/n)} \quad \Rightarrow \quad f(x) = \sqrt{x - 2}$$

and the integral would be $\int_3^6 \sqrt{x - 2}\, dx$.

As a last option, suppose $a = -1$. Then we would have

$$f(-1 + 3i/n) = \sqrt{1 + 3i/n} = \sqrt{2 + (-1 + 3i/n)} \quad \Rightarrow \quad f(x) = \sqrt{x + 2}$$

and the integral would be $\int_{-1}^{2} \sqrt{x + 2}\, dx$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + 3 \cdot \frac{25i^2}{n^2}\right) \left(\frac{5}{n}\right)$

Some options:

$$\int_0^5 2 + 3x^2\, dx \qquad \int_1^6 2 + 3(x - 1)^2\, dx \qquad \int_2^7 2 + 3(x - 2)^2\, dx$$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(3 + \frac{2i}{n}\right) \left(\frac{2}{n}\right)$

Some options:

$$\int_0^2 \sin(3 + x)\, dx \qquad \int_3^5 \sin(x)\, dx \text{ etc.}$$