Hints for Section 7.3

1. With the suggestion $x = 2\sin(\theta)$:

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\sin(\theta)\,d\theta}{(4\sin^2(\theta))(2\cos(\theta))} = \frac{1}{4}\int \csc^2(\theta)\,d\theta = -\frac{1}{4}\cot(\theta) + C$$

The substitution $\sin(\theta) = x/2$ gives us the relationship between lengths in a right triangle (write one angle as θ, the side opposite θ has length x, the hypotenuse has length 2, and the other leg has length $\sqrt{4-x^2}$ by the Pythagorean Theorem. Read off the values for $\cot(\theta)$, and you should get the answer in the back of the text.

4. This problem can actually be done faster using $u = 1 - x^2$, but let's do the trig substitution. The expression $1 - x^2$ suggests that we use $x = \sin(\theta)$. As we perform the substitution, we may wish to change the integral bounds at the same time:

$$\int x^3\sqrt{1-x^2}\,dx = \int \sin^3(\theta)\sqrt{1-\sin^2(\theta)}\,\cos(\theta)d\theta = \int \sin^3(\theta)\cos^2(\theta)d\theta$$

This last integral is true as long as $\cos(\theta) > 0$, so that $\sqrt{\cos^2(\theta)} = \cos(\theta)$. To check this, look at the integral bounds:

- If $x = 0$, then $\sin(\theta) = 0$, so that $\theta = 0$.
- If $x = 1$, then $\sin(\theta) = 1$, so that $\theta = \pi/2$.
- The cosine is positive for $0 \le \theta \le \pi/2$, so we continue with these bounds.

Now we proceed with the technique from 7.2, since we have an odd power of sine. That is, reserve one of them for u, du substitution with $u = \cos(\theta)$ (watch the integral bounds!)

$$\int_0^{\pi/2} \sin^2(\theta)\cos^2(\theta)(\sin(\theta)d\theta = \int_1^0 (1-u^2)u^2(-\,du) =$$

$$\int_0^1 u^2 - u^4\,du = \frac{1}{3}u^3 - \frac{1}{5}u^5 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

5. The expression $t^2 - 1$ suggests that we use $t = \sec(\theta)$, so that $t^2 - 1$ will simplify.

$$\int_{\sqrt{2}}^2 \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int \frac{1}{\sec^3(\theta)\sqrt{\sec^2(\theta)-1}}\,\sec(\theta)\tan(\theta)d\theta = \int \cos^2(\theta)\,d\theta$$

For the integral bounds, use the 45-45-90 and 30-60-90 triangles:

$$t = \sqrt{2} \quad \Rightarrow \quad \sec(\theta) = \sqrt{2} \quad \Rightarrow \quad \cos(\theta) = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

$$t = 2 \quad \Rightarrow \quad \sec(\theta) = 2 \quad \Rightarrow \quad \cos(\theta) = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

Continuing, use the half-angle formula to integrate:

$$\frac{1}{2}\int_{\pi/4}^{\pi/3} 1+\cos(2\theta)\,d\theta = \left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right|_{\pi/3}^{\pi/4}$$

and so on.

11. The expression $1-4x^2$ suggests that we use $2x = \sin(\theta)$.

$$\int \sqrt{1-4x^2}\,dx = \int \sqrt{1-\sin^2(\theta)}\cdot\frac{1}{2}\cos(\theta)d\theta = \frac{1}{2}\int \cos^2(\theta)\,d\theta =$$

$$\frac{1}{4}\int 1+\cos(2\theta)\,d\theta = \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C$$

Now we use $2x = \sin(\theta)$ to do our back substitution. We get $\theta = \sin^{-1}(\theta)$ and for $\sin(2\theta)$, use the identity

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

and the right triangle (one angle is θ, with opposite side length $2x$, and hypotenuse 1), so that in terms of x,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\cdot 2x\cdot\sqrt{1-4x^2}.$$

The overall solution is therefore

$$\frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2}$$

13. The expression x^2-9 suggests that we use $x = 3\sec(\theta)$, so that x^2-9 becomes $9(\sec^2(\theta)-1) = 9\tan^2(\theta)$. With that, we have:

$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan(\theta)}{27\sec^3\theta}\cdot 3\sec(\theta)\tan(\theta)\,d\theta = \frac{1}{3}\int \frac{\tan^2(\theta)}{\sec^2(\theta)}\,d\theta = \frac{1}{3}\int \sin^2(\theta)d\theta$$

Integrating this with the half-angle formula:

$$\frac{1}{6}\int 1-\cos(2\theta)d\theta = \frac{1}{6}\theta - \frac{1}{12}\sin(2\theta) + C$$

Don't use the inverse secant when converting back (only inverse sine, cosine and tangent if you can do it). That's because we should never need it, and most computers/calculators will not have an "inverse secant" button. Then we have:

$$x = 3\sec(\theta) \quad\Rightarrow\quad \cos(\theta) = \frac{3}{x} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{3}{x}\right)$$

For $\sin(2\theta)$, first write it using the trig identity, then use the triangle.

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\cdot\frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x} = \frac{6\sqrt{x^2-9}}{x^2}$$

That gives us the solution:

$$\frac{1}{6}\cos^{-1}\left(\frac{3}{x}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

15. Let $x = a \cdot \sin(\theta)$. Be careful of the integral bounds-

$$x = 0 \quad \Rightarrow \quad \sin(\theta) = 0 \quad \Rightarrow \quad \theta = 0$$

$$x = a \quad \Rightarrow \quad a\sin(\theta) = a \quad \Rightarrow \quad \theta = \pi/2$$

Now we form the integral:

$$\int_0^a x^2\sqrt{x^2 - a^2}\,dx = \int_0^{\pi/2} a^2\sin^2(\theta)\sqrt{a^2(\sin^2(\theta) - 1)} \cdot a\,\cos(\theta)d\theta = a^4\int_0^{\pi/2} \sin^2(\theta)\cos^2(\theta)\,d\theta$$

There are several ways to continue. One way is to use the half angle formula for sine and cosine, then multiply the result out. Another way is to use $\sin(2x) = 2\sin(x)\cos(x)$ and integrate:

$$\int \left(\frac{1}{2}\sin(2\theta)\right)^2 d\theta = \frac{1}{4}\int \frac{1}{2}(1 - \cos(4\theta))\,d\theta$$

19. Let $x = \tan(\theta)$:

$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sqrt{1+\tan^2(\theta)}}{\tan(\theta)} \cdot \sec^2(\theta)\,d\theta = \int \frac{\sec^3(\theta)}{\tan(\theta)}\,d\theta$$

It may not be clear what what to do from here. Hint: $\sec^2(\theta) = \tan^2(\theta) + 1$. Using that,

$$\int \frac{\tan^2(\theta)+1}{\tan(\theta)}\sec(\theta)\,d\theta = \int \sec(\theta)\tan(\theta) + \frac{\cos(\theta)}{\sin(\theta)} \cdot \frac{1}{\cos(\theta)}\,d\theta = \int \sec(\theta)\tan(\theta) + \csc(\theta)\,d\theta$$

(Remember the table of integrals you'll have). Evaluating, we have:

$$\sec(\theta) + \ln|\csc(\theta) - \cot(\theta)| + C$$

Convert this back to x using the triangle suggested by $x = \tan(\theta)$:

$$\sqrt{1-x^2} + \ln\left|\frac{\sqrt{1-x^2}}{x} - \frac{1}{x}\right| + C$$

23. Complete the square so that:

$$-x^2 + 4x + 5 = -(x^2 - 4x + \qquad) + 5 = -(x^2 - 4x + 4) + (5 + 4) = 9 - (x-2)^2.$$

Now we can substitute $x - 2 = 3\sin(\theta)$ so that:

$$\int \sqrt{5 + 4x - x^2}\,dx = \int \sqrt{9 - (x-2)^2}\,dx = \int \sqrt{9\cos^2(\theta)} \cdot 3\cos(\theta)\,d\theta = 9\int \cos^2(\theta)\,d\theta$$

Integrate this using the half angle identity, then convert back to x:

$$\frac{9}{2}\int 1 + \cos(2\theta)\,d\theta = \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) = \frac{9}{2}\theta + \frac{9}{4}2\sin(\theta)\cos(\theta)$$

Converting back and simplifying, we get what's in the back of the text.

25. Complete the square so that

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \quad \Rightarrow \quad x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan(\theta)$$

The integral simplifies to integrating $\tan(\theta)\sec(\theta)$ and $\sec(\theta)$.

27. Complete the square so that $x^2 + 2x = (x+1)^2 - 1$, and let $x + 1 = \sec(\theta)$. We end up integrating $\tan^2(\theta)\sec(\theta)$ (write all in terms of $\sec(\theta)$ and use the table.

29. Let $u = x^2$ and $du = 2x\,dx$. Then let $u = \sin(\theta)$ (or do the substitution directly by taking $x^2 = \sin(\theta)$, etc. End up integrating $\cos^2(\theta)$.

31(a) Let $x = a\tan(\theta)$, and end up integrating $\sec(\theta)$.

33. **Typo: Ignore this problem until we've gone through 6.5**.

35. The area of the triangle POQ is

$$\frac{1}{2}(r\cos(\theta))(r\sin(\theta)) = \frac{1}{2}r^2\cos(\theta)\sin(\theta)$$

The area of PQR is

$$\int_{r\cos(\theta)}^{r} \sqrt{r^2 - x^2}\,dx$$

We find that the area of the desired sector is the sum of the area of the triangle and the area using the integral above. Summing these, we get $\frac{1}{2}r^2\theta$.