Homework Hints, Section 7.4

5. Do long division in part (a), then factor the denominator to end up with

$$x^4 + 4x^2 + 16 + \frac{A}{x+2} + \frac{B}{x-2}$$

In part (b), note that $x^2 - x + 1$ is irreducible, so you have three terms in the sum.

7. Do long division.

15. Long division first.

19. We end up needing to solve:

$$x^2 + 1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$$

Try letting $x = 2$, then $x = 3$, then equate the coefficients of x^2.

21. Long division gives us:

$$\frac{x^3+4}{x^2+4} = x + \frac{4-4x}{x^2+4}$$

The second term should be broken up when integrating:

$$\int \frac{4-4x}{x^2+4}\,dx = \int \frac{4}{x^2+4}\,dx - 2\int \frac{2x}{x^2+4}\,dx$$

For the first integral, we can use trig substitution with $x = 2\tan(\theta)$. Continuing,

$$\int \frac{4}{x^2+4}\,dx = \int \frac{4(2\sec^2(\theta)\,d\theta)}{4\tan^2(\theta)+4} = \int \frac{8\sec^2(\theta)}{4\sec^2(\theta)}\,d\theta = 2\theta = 2\tan^{-1}(x/2)$$

For the other integral, use $u = x^2 + 4$ so that $du = 2x\,dx$, and the integral becomes

$$\int \frac{2x\,dx}{x^2+4} = \int \frac{1}{u}\,du = \ln|x^2+4|$$

For the overall answer, sum the three expressions (and add C!).

23. Do the partial fraction decomposition, and find that

$$\frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} - \frac{x+1}{x^2+9}$$

The integral is similar to #21, where we break up the fraction with $x^2 + 9$:

$$\int \frac{x+1}{x^2+9}\,dx = \int \frac{x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx$$

The first integral is computed using $u = x^2 + 9$, and the second is using $x = 3\tan(\theta)$.

24. We'll do this one out, since it's an even problem.

First, set up the partial fraction decomposition:

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

Multiply both sides by the denominator, and solve for A, B, C:

$$x^2 - x + 6 = A(x^2+3) + (Bx+C)x$$

If $x = 0$, then $6 = 3A$, or $A = 2$. Continuing, we might substitute numbers in for x or equate coefficients:

$$x^2 - x + 6 = 2x^2 + 6 + Bx^2 + Cx$$

The coefficients for x^2 are equal: $1 = 2 + B$, so $B = -1$.

The coefficients for x are equal: $-1 = C$

Now we can integrate:

$$\int \frac{2}{x}\,dx - \int \frac{x}{x^2+3}\,dx - \int \frac{1}{x^2+3}\,dx$$

Now we do each one separately. The first is easy- $\int 2/x\,dx = 2\ln|x|$. Continuing as in 21 and 23, we use $u = x^2 + 3$ in the next one (and trig substitution in the last):

$$\int \frac{x}{x^2+3}\,dx = \int \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\ln|x^2+3|$$

Next is trig substitution with $x = \sqrt{3}\tan(\theta)$:

$$\int \frac{1}{x^2+3}\,dx = \frac{\sqrt{3}\sec^2(\theta)\,d\theta}{3\sec^2(\theta)} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

Put them together (remember the negative signs from the partial fractions):

$$2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

35. In this one, we'll focus on the algebra needed for the constants- The integrals can all be done with u, du substitution:

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

Now we have:

$$1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x$$

so if $x = 0$, we have $A = 1/16$. From here, we might put in the value of A, then equate coefficients to the remaining polynomials:

$$1 = \frac{1}{16}(x^4 + 8x^2 + 16) + Bx^4 + Cx^3 + (4B + D)x^2 + (4C + E)x$$

Now, equate the coefficients:

- x^4 terms: $0 = \frac{1}{16} + B$, so $B = -1/16$
- x^3 terms: $0 = C$
- x^2 terms: $0 = \frac{1}{2} + (4B + D)$. With $B = -1/16$, we get $D = -1/4$.
- x terms: $0 = (4C + E)$, and with $C = 0$, then $E = 0$ as well.
- The constant terms already work out.

Now we integrate:

$$\frac{1/16}{x} + \frac{-(1/16)x}{x^2 + 4} + \frac{-(1/4)x}{(x^2 + 4)^2}$$

The first one is ready to go, the other two both use $u = x^2 + 4$.

39. Let $u = \sqrt{x + 1}$ so that $u^2 = x + 1$ and $2u\,du = dx$. The integral then changes to

$$\int \frac{2u^2}{u^2 - 1}\,du$$

(Then do partial fractions on that, but remember to do long division first!)

41. A little tricky, but if we let $u = \sqrt{x}$, then $u^2 = x$, and $2u\,du = dx$. Making these substitutions, we get

$$\int \frac{dx}{x^2 + x\sqrt{x}} = \int \frac{2u\,du}{u^4 + u^2 \cdot u} = \int \frac{2u\,du}{u^3(u + 1)} = \int \frac{2du}{u^2(u + 1)}$$

Now partial fractions. You should find that:

$$\frac{2}{u^2(u + 1)} = \frac{-2}{u} + \frac{2}{u^2} + \frac{2}{u + 1}$$

so that each is easily integrated.

47. It's clear that we want to substitute $u = \mathrm{e}^x$. What might not be clear is what to do after. Taking the logarithm of both sides, we can solve for x:

$$x = \ln(u) \quad \Rightarrow \quad dx = \frac{1}{u}\,du$$

And now, with the substitutions:

$$\int \frac{\mathrm{e}^{2x}\,dx}{\mathrm{e}^{2x} + 3\mathrm{e}^x + 2} = \int \frac{u^2(du/u)}{u^2 + 3u + 2} = \int \frac{u\,du}{u^2 + 3u + 2} = \int \frac{u\,du}{(u + 1)(u + 2)}$$

At this point, proceed with the usual partial fraction decomposition.

57. Complete the square in the denominator:

$$\int \frac{dx}{x^2 - 2x} = \int \frac{dx}{x^2 - 2x + 1 - 1} = \int \frac{dx}{(x-1)^2 - 1} = \int \frac{du}{u^2 - 1} = \int \frac{du}{(u+1)(u-1)}$$

And proceed with partial fractions.

58. Complete the square in the denominator:

$$4x^2 + 12x - 7 = 4\left((x^2 + 3x + \frac{9}{4}\right) - 7 - 9 = [2(x + 3/2)]^2 - 16 = (2x+3)^2 - 16$$

One way to go is to let $u = 2x + 3$, so the numerator is $2x + 1 = u - 2$, and you could write:

$$\int \frac{2x+1}{4x^2 + 12x - 7}\, dx = \frac{1}{2} \int \frac{u-2}{u^2 - 16}\, du = \int \frac{u-2}{(u+4)(u-4)}\, du$$

and proceed with partial fractions.

59. This one is for your own amusement.

60. This won't be on the exam, but as a hint, take $t = \tan(x/2)$, then use the results of 59.