Homework Hints, Section 11.1

2. A convergent sequence is a sequence for which the limit exists. Here are two examples. The first is has a limit at zero, the second at $\pi/2$ (the horizontal asymptote for the inverse tangent function).

$$a_n = \frac{n^2}{e^n} \qquad qquad \qquad a_n = \tan^{-1}(n)$$

Here are two examples of divergent sequences. One increases without bound, and the other oscillates between two numbers.

$$a_n = \frac{e^n}{n^3} \qquad\qquad a_n = (-1)^n$$

6. If $a_n = \cos(n\pi/2)$, then you can read the sequence off the graph of the cosine function (if the beginning index is not given, you may assume it begins with $n = 1$).

$$\{0, -1, 0, 1, 0, -1, \ldots\}$$

8. $a_n = (-1)^n \frac{n}{n!+1}$
Nothing special here, this was assigned for practice with the factorial $n!$.

$$\left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{7}, \frac{4}{25}, -\frac{5}{121} \ldots\right\}$$

9. This is called a recursive sequence. Normally, $a_n = f(n)$, but in this case, we have the next term as a function of the previous terms. The difference between these is that with $a_n = f(n)$, we can compute the value of the sequence for any n without computing anything else. For a recursive sequence, you need to compute the sequence all the way up to n. In this case,

$$a_1 = 1, \quad a_2 = 5(1) - 3 = 2 \quad a_3 = 5(2) - 3 = 7,$$

and so on.

13. Problems 13 and 15 are there for you to practice looking for patterns.

19. Problems 19 and 21 are there for you to see how to associate the limit with computation. These are straightforward enough that if you want to simply compute the limit, that's fine.

25. Use L'Hospital's rule, or divide numerator and denominator by n^2.

28. This was assigned for the algebraic manipulation:

$$\frac{3^{n+2}}{5^n} = \frac{3^2 3^n}{5^n} = 9\left(\frac{3}{5}\right)^n$$

Since $3/5$ is less than 1, as $n \to \infty$, $a_n \to 0$.

29. Think about the general rule:

$$\lim_{n\to\infty} g(a_n) = g\left(\lim_{n\to\infty} a_n\right) = g(L)$$

which is true if g is continuous at L (the limit of the sequence). In this particular case, let

$$a_n = \frac{2\pi n}{1+8n}$$

As $n \to \infty$, $a_n \to \frac{2\pi}{8} = \frac{\pi}{4}$, and the tangent function is continuous at $\pi/4$ (its horizontal asymptotes are at multiples of $\pi/2$). Therefore,

$$\lim_{n\to\infty} \tan\left(\frac{2\pi n}{1+8n}\right) = \tan\left(\lim_{n\to\infty} \frac{2\pi n}{1+8n}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

33. Consider $|a_n|$ to get rid of $(-1)^n$, then it is easy to compute the limit:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$$

therefore, $a_n \to 0$ as well.

36. Same idea as #29, the limit is $\cos(0) = 1$.

50. Use l'Hospital's rule:

$$\lim_{n\to\infty} \frac{(\ln(n))^2}{n} = \lim_{n\to\infty} \frac{2\ln(n)\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2\ln(n)}{n} = \lim_{n\to\infty} \frac{2\frac{1}{n}}{1} = 0$$

55. As a hint, write out the factorial first:

$$\frac{n!}{2^n} = \frac{1\cdot 2\cdot 3\cdot 4\cdots n}{2\cdot 2\cdot 2\cdot 2\cdots 2}$$

Notice that as n increases, we are multiplying by numbers that are larger than 1, and getting bigger and bigger:

$$\frac{n!}{2^n} = \frac{1}{2}\cdot 1\cdot \frac{3}{2}\cdot 2\cdot \frac{5}{2}\cdot 3\cdots \frac{n}{2}$$

so we see that a_n is diverging (to infinity). A formal way of saying this would be to observe that

$$\frac{n!}{2^n} = \frac{1\cdot 2\cdot 3\cdot 4\cdots n}{2\cdot 2\cdot 2\cdot 2\cdots 2} > \frac{1}{2}\frac{n}{2} = \frac{1}{4}n$$

and this smaller sequence diverges (so the larger sequence must also diverge).

59. This one uses the same technique as #29 and #36:

$$\lim_{n\to\infty} \sqrt{\frac{3+2n^2}{8n^2+n}} = \sqrt{\lim_{n\to\infty} \frac{3+2n^2}{8n^2+n}} = \sqrt{\frac{2}{8}} = \frac{1}{2}$$

62. This one gives us some practice dealing with sequences involving factorials. To get a feeling for what this sequence is, let's compute the first few terms:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{1 \cdot 3}{2} = \frac{3}{2} \\ a_3 &= \frac{1 \cdot 3 \cdot 5}{2 \cdot 2} > \frac{3 \cdot 3}{2 \cdot 2} \\ a_4 &= \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 2} > \frac{3 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 2} \end{aligned}$$

From this, we conclude that $a_n > (3/2)^{n-1}$, which diverges. Therefore, a_n diverges as well.

73. The sequence is decreasing, since

$$a_{n+1} = \frac{1}{2(n+1)+3} < \frac{1}{2n+3} = a_n$$

75. The sequence is neither increasing nor decreasing- it is "oscillating" from the $(-1)^n$ term (and going to $\pm\infty$).