Homework Hints, 11.10- Taylor and Maclaurin Series

1-4, 5, 6, 9, 13-14, 16, (25, 27)*, (29, 31, 33)**, 39***, 56-57, 63, 65, 66, 68
where $*$−Use Theorem 17, (**) means to use Table 1, and (***) means to use a computer or graphing calculator.

1. The question is a little vague- It should say: "... write a formula for b_8 in terms of f." Then we just have the formula from the Taylor series:

$$f(x) = b_0 + b_1(x-5) + b_2(x-5)^2 + \cdot + b_8(x-5)^8 + \cdots$$

so that differentiating 8 times gives us:

$$f^{(8)}(x) = 8!b_8 + 9!b_9(x-5) + \cdots \quad \Rightarrow \quad b_8 = \frac{f^{(8)}(5)}{8!}$$

2(a). It looks like $f(1) > 1$, so its possible that $c_0 = 1.6$.

Going to the next term, $1.6 - 0.8(x-1)$ should be $f(1) + f'(1)(x-1)$ (the tangent line approximation to f at $x = 1$). That would mean that $f'(1) = -0.8$, but f is increasing at 1, which means $f'(1) > 0$. Therefore, this cannot be the Taylor series for f.

2(b). Similar in argument to 2(a), at $x = 2$, we should have:

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots = 2.8 + 0.5(x-2) + 1.5(x-2)^2 + \cdots$$

It is plausible that $f(2) \approx 2.8$ and $f'(2) \approx 0.5$, but what about $\frac{f''(2)}{2} \approx 1.5$? It is apparent from the graph of the function that f is concave down at $x = 2$, which means that $f''(2) < 0$. But this would contradict our earlier statement, that $f''(2) \approx 3$.

3. The problem statement is telling us that:

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{(n+1)!}{n!} = n+1 \quad \Rightarrow \quad \sum_{n=0}^{\infty}(n+1)x^n$$

Use the Ratio Test to finish.

4. The problem statement is telling us how to get c_n for the Taylor series:

$$c_n = \frac{f^{(n)}(4)}{n!} = \frac{(-1)^n n!}{3^n(n+1)} \cdot \frac{1}{n!} = \frac{(-1)^n}{3^n(n+1)}$$

Therefore, the Taylor series for this function is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-4)^n$$

For the radius of convergence, use the Ratio Test, and find that $\rho = 3$.

For 5, 6 and 9, try to find the pattern in the derivatives

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
$n=0$	$(1-x)^{-1}$	1
$n=1$	$(1-x)^{-2}$	1
$n=2$	$2(1-x)^{-3}$	2
$n=3$	$3\cdot 2(1-x)^{-4}$	$3!$
$n=4$	$4\cdot 3\cdot 2(1-x)^{-5}$	$4!$
$\vdots$		
n		$n!$

$$\Rightarrow \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n$$

which of course converges for $|x| < 1$ (it is a geo series).

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
$n=0$	$\ln(1+x)$	0
$n=1$	$(1+x)^{-1}$	1
$n=2$	$-(1+x)^{-2}$	-1
$n=3$	$2(1-x)^{-3}$	2
$n=4$	$-3\cdot 2(1-x)^{-4}$	$-3!$
$n=5$	$4\cdot 3\cdot 2(1-x)^{-5}$	$4!$
$\vdots$		
n		$(-1)^{n+1}(n-1)!$

$$\Rightarrow \quad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$$

which converges for $|x| < 1$ (by the Ratio Test).

9.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
$n=0$	2^x	1
$n=1$	$2^x ln(2)$	$\ln(2)$
$n=2$	$2^x(\ln(2))^2$	$(\ln(2))^2$
$n=3$	$2^x(\ln(2))^3$	$(\ln(2))^3$
$\vdots$		
n		$(\ln(2))^n$

$$\Rightarrow \quad 2^x = \sum_{n=1}^{\infty} \frac{(\ln(2))^n}{n!} x^n$$

By the Ratio Test, we see that the series converges for all x.

14. (This is very similar to #13)

$n=0$	$x-x^3$	$-2+8=6$
$n=1$	$1-3x^2$	-11
$n=2$	$-6x$	12
$n=3$	-6	-6

$$\Rightarrow \quad x - x^3 = 6 - 11(x+2) + \frac{12}{2}(x+2)^2 - \frac{6}{6}(x+2)^3$$

We should simplify this to: $6 - 11(x+2) + 6(x+2)^2 - (x+2)^3$, but no further! If we were to expand this out, we would end up where we started: $x - x^3$. In #13 and #14, the radius of convergence is ∞, since it is ALWAYS possible to sum four terms.

16. Similar to 5, 6, 9. We want to see some pattern in the derivatives, then find a formula for c_n. After that, we could use the Ratio Test to determine the radius of convergence.

n	$f^{(n)}(x)$	$f^{(n)}(-3)$
$n=0$	x^{-1}	$(-3)^{-1} = -3^{-1}$
$n=1$	$-x^{-2}$	$-(-3)^{-2} = -3^{-2}$
$n=2$	$2x^{-3}$	$2\cdot(-3)^{-3} = -2\cdot 3^{-3}$
$n=3$	$-3\cdot 2x^{-4}$	$-3!\cdot 3^{-4}$
$n=4$	$4\cdot 3\cdot 2x^{-5}$	$-4!\cdot 3^{-5}$
$\vdots$		
n		$-n!\cdot 3^{-(n+1)}$

There is our formula for $f^{(n)}(-3)$, so putting that into the Taylor Series, we get:

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{-n!}{n!\cdot 3^{n+1}}(x+3)^n = -\sum_{n=0}^{\infty} \frac{(x+3)^n}{3^{n+1}}$$

To find the radius of convergence, use the Ratio Test to see that $\rho = 3$.

Exercises #25, 27 are to practice the Binomial Theorem, but we can also do them from scratch. You won't need to have the Binomial Theorem memorized for the exam.

25. Here is a table of the derivatives we get for this function. For clarity, we put the values of $4n-5$ in the last column- That was found to be the number we stop at in the numerator. The exceptions to this is when $n = 0, 1$, so we'll split those off.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$4n-5$
0	$(1-x)^{1/4}$	1	
1	$-\frac{1}{4}(1-x)^{-3/4}$	$-\frac{1}{4}$	
2	$-\frac{3}{4^2}(1-x)^{-7/4}$	$-\frac{3}{4^2}$	3
3	$-\frac{7\cdot 3}{4^3}(1-x)^{-11/4}$	$-\frac{3\cdot 7}{4^3}$	7
4	$-\frac{11\cdot 7\cdot 3}{4^4}(1-x)^{-15/4}$	$-\frac{3\cdot 7\cdot 11}{4^4}$	11

Now we can write the series:

$$1 - \frac{1}{4}x - \sum_{n=2}^{\infty} \frac{3\cdot 7\cdot 11\cdots(4n-5)}{4^n n!}x^n$$

27. Do the same as #25, and you'll find the pattern to be the following. Notice that a 2 was missing from the factorials, that's what the fourth column corrects for:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	
0	$(2+x)^{-3}$	2^{-3}	$2 \cdot 2^{-3} \cdot (1/2)$
1	$-3(2+x)^{-4}$	$-3 \cdot 2^{-4}$	$-3! \cdot 2^{-4}(1/2)$
2	$4 \cdot 3(2+x)^{-5}$	$4 \cdot 3 \cdot 2^{-5}$	$4! \cdot 2^{-5}(1/2)$
3	$5 \cdot 4 \cdot 3(2+x)^{-6}$	$-5 \cdot 4 \cdot 3 \cdot 2^{-6}$	$5! \cdot 2^{-6}(1/2)$

$$\Rightarrow \qquad f^{(n)}(0) = (-1)^n \frac{(n+2)!}{2^{n+4}}$$

Therefore, the series becomes the following (remember to divide by $n!$):

$$\frac{1}{(2+x)^3} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2^{n+4}} x^n$$

For Exercises 29-33, use Table 1. We'll do 33 in detail below, the others are similar.

33. We want the Maclaurin series for $f(x) = x\cos(x^2/2)$. We start with the series for $\cos(x)$:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

so that

$$\cos\left(\frac{x^2}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{x^2}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!2^{2n}} x^{4n}$$

Finally, multiply both sides by x:

$$x\cos\left(\frac{x^2}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!2^{2n}} x^{4n+1}$$

The radius of convergence was ∞, and that hasn't changed with our transformation.

39. This is a graphing exercise (which won't be on the exam), however, you should be able to compute the Maclaurin series quickly (similar to #33):

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} = 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 + \cdots$$

You should then plot $\cos(x^2)$ two, then three terms to see that the more terms you add, the closer the polynomial comes to looking like $\cos(x^2)$.

56. Use the series expansions for $\cos(x)$ and e^x. For example, the numerator of our expression is:

$$1 - \cos(x) = 1 - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots)$$

Simplifying, the numerator is:

$$\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots$$

Looking at the denominator, we have:

$$1 + x - e^x = 1 + x - (1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots) = -\frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \cdots$$

Divide numerator and denominator by x^2 to get:

$$\frac{1 - \cos(x)}{1 + x - e^x} = \frac{\frac{1}{2!} - \frac{1}{4!}x^2 + \frac{1}{6!}x^4 + \cdots}{-\frac{1}{2!} - \frac{3!}{x} - \frac{1}{4!}x^2 + \cdots}$$

Therefore, taking the limit as $x = 0$, we get -1.

57. Similar to 56, we should get that

$$\frac{\sin(x) - x + \frac{1}{6}x^3}{x^5} = \frac{1}{5!} - \frac{1}{7!}x^2 + \frac{1}{9!}x^4 - \cdots$$

For #63-66, recognize the sums as the Taylor/Maclaurin series of some template function.

63. Recognize this one as the exponential:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so that

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}$$

65. This one is similar to the last one. Go to Table 1 first and see if any of the functions listed there will work. The only one there with n alone in the denominator is the series for $\ln(1 + x)$, so we use that one:

$$\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

This matches the textbook problem if $x = 3/5$- Therefore,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(3/5)^n}{n} = \ln\left(1 + \frac{3}{5}\right) = \ln(8/5)$$

66. We're back to the exponential:

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!} = \mathrm{e}^{3/5}$$

68. In this one, the authors are trying to hide the pattern by expanding the series. If we write the series in sigma notation:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(\ln(2))^n}{n!} = \sum_{n=0}^{\infty} \frac{(-\ln(2))^n}{n!} = \mathrm{e}^{-\ln(2)} = \frac{1}{2}$$