Homework Hints, Section 11.2

1, 4, 5, 9, 14, 16 23, 28-30, 35, 42, 55, 57, 58

4. Recall that the infinite sum is the limit of the partial sums, s_n:

$$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n^2-1}{4n^2+1} = \frac{1}{4}$$

5. This exercise is good to make sure we understand the notation, and to get some numerical experience with the limit. Here is a table with the values:

Partial Sum	Value
S_1	1.000000000000000
S_2	1.125000000000000
S_3	1.162037037037037
S_4	1.177662037037037
S_5	1.185662037037037
S_6	1.190291666666667
S_7	1.193207118561710
S_8	1.195160243561710

It appears, as n increases, the value of S_n is approaching some number.

9. Similar to #5, here is a table with the partial sums:

S_1	-2.400000000000000
S_2	-1.920000000000000
S_3	-2.016000000000000
S_4	-1.996800000000000
S_5	-2.000640000000000
S_6	-1.999872000000000
S_7	-2.000025600000000
S_8	-1.999994880000000
S_9	-2.000001024000000
S_{10}	-1.999999795200000

It appears that as $n \to \infty$, our values for S_n are approaching -2. We can show this, since this is a Geometric Series:

$$\sum_{n=1}^{\infty} 12\left(-\frac{1}{5}\right)^n = \sum_{n=1}^{\infty} ar^n = \frac{(-12/5)}{1-(-1/5)} = \frac{-12/5}{-6/5} = \frac{-12}{5}\cdot\frac{5}{6} = -2$$

14. This is similar to #9:

S_1	0.125000000000000
S_2	0.191666666666667
S_3	0.233333333333333
S_4	0.261904761904762
S_5	0.282738095238095
S_6	0.298611111111111
S_7	0.311111111111111
S_8	0.321212121212121
S_9	0.329545454545455

It appears again that S_n approaches a limit. This is more difficult to show analytically, but is almost identical to **Example 7**. We use partial fractions to find the limit. The partial fraction step is:

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \quad \Rightarrow \quad A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

Therefore,

$$\sum_{n=2}^{k} \frac{1}{n(n+2)} = \frac{1}{2} \sum_{n=2}^{k} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

Following that example, if we write down the terms of the sum, we get:

$$= \frac{1}{2} \left[\left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \cdots + \left(\frac{1}{k-2} - \frac{1}{k} \right) + \left(\frac{1}{k-1} - \frac{1}{k+1} \right) + \left(\frac{1}{k} - \frac{1}{k+2} \right) \right]$$

And everything cancels except the following:

$$\sum_{n=2}^{k} \frac{1}{n(n+2)} = \frac{1}{2} \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{k+1} - \frac{1}{k+2} \right]$$

Now, in the limit as $k \to \infty$, we end up with $(1/2) \cdot (5/6) = 5/12$.

16. The first two sums use a different index, but represent the exact same sum. In part (b), the first one is correct, the second one is not (the index does not match the subscript).

23. This is a really important example, so be sure you understand it! Rewrite the sum so that it looks like a geometric series, then use that to determine the limit.

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{(-3)^{-1}(-3)^n}{4^n} = -\frac{1}{3} \sum_{n=1}^{\infty} \left(-\frac{3}{4} \right)^n = -\frac{1}{3} \cdot \frac{-3/4}{1-(-3/4)} = \frac{1}{7}$$

28. In this one, you might break up the sum in an easier way:

$$\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \cdots = \left(\frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \cdots \right) + 2 \left(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots \right)$$

The odd numbers $(1,3,5,\cdots)$ are given by $2n-1$, $n=1,2,3,\cdots$, and the even numbers are given by $2n$, $n=1,2,3,\cdots$. These can now be written as:

$$\sum_{n=1}^{\infty}\frac{1}{3^{2n-1}}+2\sum_{n=1}^{\infty}\frac{1}{3^{2n}}$$

This is a sum of two convergent geometric series, so overall we get a convergent series. We'll compute the sums separately, then add them at the end:

$$\sum_{n=1}^{\infty}\frac{1}{3^{2n-1}}=\sum_{n=1}^{\infty}\frac{1}{3^{2n}3^{-1}}=3\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n=\frac{3/9}{1-(1/9)}=\frac{3}{8}$$

And similarly

$$\sum_{n=1}^{\infty}\frac{1}{3^{2n}}=\sum_{n=1}^{\infty}\frac{1}{(3^2)^n}=\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n=\frac{1/9}{1-(1/9)}=\frac{1}{8}$$

The overall series then converges to:

$$\frac{3}{8}+2\cdot\frac{1}{8}=\frac{5}{8}$$

29. Diverges by the Test for Divergence.

30. Diverges by the Test for Divergence.

35. Diverges by the Test for Divergence.

42. Diverges by the Test for Divergence.

55. This is a nice exercise that mimics the technique we used for the sum of a geometric series. First, if $S=1.53424242...$, then we multiply by 100 so that only the repeating part is left after the decimal. Multiply both sides of that by 100 to get another number with the same repeating part, then subtract those:

$$\begin{array}{rl} -100S & =-153.424242... \\ 10000S & =15342.424242... \\ \hline 9900S & =15189 \end{array} \quad \Rightarrow \quad S=\frac{15189}{9900}$$

57. That's a geometric series with $r=-5x$, so the sum is

$$\frac{-5x}{1+5x}$$

It converges as long as $|r|<1$, or in this case, $|-5x|<1$, or $|x|<1/5$.

58. Same idea as #57, with $r=x+2$, so the sum is

$$\frac{x+2}{1-(x+2)}=-\frac{x+2}{x+1}$$

which converges as long as $|x+2|<1$, or $-1<x+2<1$ which gives $-3<x<-1$