Homework hints: Section 11.3

3,6,13,15,19,21,27,30,32

3. Be sure to verify that $x^{-1/5}$ is positive, continuous and decreasing for $x > 0$.

6. Since this is an even numbered problem, we show the solution below. To use the integral test, we see that
$$f(x) = (x+4)^{-1/2}$$
This is a positive, continuous function for $x \neq -4$. It is also decreasing, since
$$f'(x) = -\frac{1}{2}(x+4)^{-3/2} = -\frac{1}{2(x+4)^{3/2}} < 0.$$
We also have:
$$\int_1^\infty (x+4)^{-1/2}\,dx = \lim_{T\to\infty} \left(\sqrt{x+4} \right|_0^T = \infty$$
Therefore, the series diverges.

13. Notice that the denominator consists of the odd integers, which are given by the formula $2n-1$, if $n = 1, 2, 3, \cdots$.

15. Hint: Split this into two series,
$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

21. Straightforward- We see that the function is decreasing since if n is replaced by $n+1$, the denominator increases (so the fraction overall decreases). To integrate, use u, du substitution.

27. If we look at the graph of the cosine function, we see that:
$$\{\cos(\pi n)\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \cdots\}$$
Therefore, $f(x) = \cos(\pi x)/\sqrt{x}$ is NOT a positive function.

30. Let
$$f(x) = \frac{1}{x\ln(x)(\ln(\ln(x)))^p}.$$
For $x \geq 3$ and $p > 0$, it is clear that f is decreasing (the denominator increases as x increases). It is also positive and continuous.

To integrate f, let $u = \ln(\ln(x))$ so that by the Chain Rule,
$$\frac{du}{dx} = \frac{1}{\ln(x)} \cdot \frac{1}{x} = \frac{1}{x\ln(x)}$$
We can also substitute for $x = 3$ ($u = \ln(\ln(3))$), and as $x \to \infty$, $u \to \infty$:
$$\int_3^\infty \frac{1}{x\ln(x)((\ln(\ln(x)))^p}\,dx = \int_{\ln(\ln(3))}^\infty u^{-p}\,du$$
We have already determined that this integral converges for $p > 1$ and diverges for $p \leq 1$.

32. The function is

$$f(x) = \frac{\ln(x)}{x^p} \quad \Rightarrow \quad f'(x) = -\frac{p\ln(x) - 1}{x^{p+1}}$$

which is negative if $p\ln(x) > 1$, or when $\ln(x) > 1/p$, or $x > e^{1/p}$. Therefore, f is eventually strictly decreasing (which is all we need for the test).

The function f is clearly positive and continuous on $[1, \infty)$ as well, so we integrate:

$$\int_1^\infty \frac{ln(x)}{x^p}\,dx \quad \Rightarrow \quad \begin{array}{ccc} + & \ln(x) & x^{-p} \\ - & 1/x & x^{-p+1}/(1-p) \end{array} \quad \Rightarrow \quad \left(\frac{x^{1-p}\ln(x)}{1-p} \right|_1^\infty + \frac{1}{1-p}\int_1^\infty x^{-p}\,dx$$

The second term (the integral) is known to converge when $p > 1$ and diverge when $p \leq 1$. Checking the first limit:

$$\lim_{T\to\infty} \frac{T^{1-p}\ln(T)}{1-P} - 0$$

This has a limit when $1 - p < 0$, or when $p > 1$. Therefore, overall, the series converges when $p > 1$.