Selected Answers: 11.4

4. Divergent; limit comparison with $1/n$

   By limit comparison:
   $$\frac{a_n}{b_n} = \frac{n^3/(n^4-1)}{1/n} = \frac{n^4}{n^4-n}$$
   so the limit (either by dividing numerator and denominator by n^4 or by L'Hospital's rule) is 1.

   By direct comparison:
   $$\frac{n^3}{n^4-1} > \frac{n^3}{n^4} = \frac{1}{n}$$

7. Convergent- Comparison with geo series, $r = 9/10$

   By limit comparison:
   $$\frac{a_n}{b_n} = \frac{9^n}{3+10^n}\frac{10^9}{9^n} = \frac{90^n}{3\cdot 9^n + 90^n}$$
   Divide numerator and denominator by 90^n and take the limit.

   By direct comparison:
   $$\frac{9^n}{3+10^n} < \frac{9^n}{10^n} = \left(\frac{9}{10}\right)^n$$

9. Convergent- Comparison with p-series, $1/n^2$

   By direct comparison:
   $$\frac{\cos^2(n)}{n^2+1} \le \frac{1}{n^2+1} < \frac{1}{n^2}$$
   (The limit would be messier)

14. Divergent- Compare with p-series, $1/\sqrt{n}$

    By direct comparison:
    $$\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$

18. Divergent by limit comparison with $1/n$

    Use the limit comparison:
    $$\frac{a_n}{b_n} = \frac{1}{2n+3}\frac{n}{1} = \frac{n}{2n+3}$$
    Either divide numerator and denominator by n or use L'Hospital's rule.

19. Divergent by limit comparison with geo series, $r = 4/3$

    Here's the algebra (in the second step, divide by 12^n)
    $$\frac{\frac{1+4^n}{1+3^n}}{\frac{4^n}{3^n}} = \frac{3^n+12^n}{4^n+12^n} = \frac{\frac{1}{4^n}+1}{\frac{1}{3^n}+1}$$

21. Convergent by limit comparison with p-series, $1/n^{3/2}$

Doing a limit comparison, here is the algebra. In the second step, divide by $n^2 = \sqrt{n^4}$:

$$\frac{n^{3/2}\sqrt{n+2}}{2n^2+n+1} = \frac{\sqrt{n^4+2n^3}}{2n^2+n+1} = \frac{\sqrt{1+\frac{2}{n}}}{2+\frac{1}{n}+\frac{1}{n^2}}$$

so the limit will be 1/2.

24. Divergent by limit comparison with harmonic $1/n$

Here's the algebra:

$$\frac{n(n^2-5n)}{n^3+n+1} = \frac{n^3-5n^2}{n^3+n+1}$$

And either divide by n^3 or use L'Hospital's rule to find the limit.

28. Divergent by comparison with harmonic series

Notice that $e^{1/n}$ is decreasing to 1 (verify with the derivative) as n increases to infinity. Therefore,

$$\frac{e^{1/n}}{n} \geq \frac{1}{n}$$

and the series with $1/n$ diverges. Therefore, the given series also diverges.

31. Use limit comparison with $1/n$ to show it diverges.

Here are the details, using L'Hospital's rule (we have a "0/0" situation)

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{n\to\infty} \frac{\cos(1/n)\cdot(-1/n^2)}{(-1/n^2)} = \cos(0) = 1$$

37. Since

$$\frac{d_n}{10^n} \leq \frac{9}{10^n} \text{ for all n}$$

and since $\sum \frac{9}{10^n}$ is a convergent series, then $\sum \frac{d_n}{10^n}$ will always converge by the comparison test.

40. Since the limit is zero, we can assume that there is a value of N so that, when $n \geq N$, a_n is smaller than b_n. Thus, this is the condition for the comparison test, and $\sum a_n$ will converge.

For the first exercise, use $b_n = 1/n^2$, and for the other, use $b_n = 1/e^n$.