Homework Hints, 11.8 (Power Series)

2, 3, 7, 10, 13, 15, 22, 24, 29, 30, 31, 33, 34

2(a). What is the radius of convergence of a power series?

SOLUTION: It is a number ρ so that, if $|x-a|<\rho$, the series converges (absolutely), and if $|x-a|>\rho$, the series diverges.

How do you find the radius of convergence?

SOLUTION: You find it by using the Ratio Test on the series.

2(b). What is the interval of convergence?

SOLUTION: It is the set of all x for which the series converges (absolute or conditional).

How is it found?

SOLUTION: Find the radius of convergence, then test the endpoints of the interval- That is, where $x=a-\rho$ and $x=a+\rho$ to see if the corresponding series converges (either kind) or diverges.

10. Find the radius and interval of convergence: $\displaystyle\sum_{n=1}^{\infty}\frac{10^n x^n}{n^3}$

SOLUTION: First use the Ratio Test to find the radius of convergence, ρ. Don't forget the absolute value signs!

$$\lim_{n\to\infty}\frac{10^{n+1}|x|^{n+1}}{(n+1)^3}\cdot\frac{n^3}{10^n|x|^n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3\cdot 10|x|=10|x|<1 \qquad\Rightarrow\qquad |x|<\frac{1}{10}$$

Therefore, the radius of convergence is $\rho=1/10$. Now check the endpoints, $\pm 1/10$:

- If $x=1/10$, then substitution into the power series gives the following, which converges.

$$\sum_{n=1}^{\infty}\frac{10^n\cdot(1/10)^n}{n^3}=\sum_{n=1}^{\infty}\frac{1}{n^3}\qquad\text{convergent } p-\text{series}$$

- If $x=-1/10$, then substitution into the power series gives the following, which converges.

$$\sum_{n=1}^{\infty}\frac{10^n\cdot(-1/10)^n}{n^3}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\qquad\text{converges absolutely}$$

Therefore, the interval of convergence includes both endpoints, and is: $\left[-\frac{1}{10},\frac{1}{10}\right]$

13. In this case, we find the radius and interval of convergence for the following, which might have some tricky algebra because of the natural log:

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{4^n \ln(n)}$$

In the Ratio Test, we should get the following, which is found by using l'Hospital's rule:

$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} \cdot \frac{|x|}{4} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{|x|}{4} = \frac{|x|}{4} < 1 \quad \Rightarrow \quad |x| < 4$$

Therefore, the radius of convergence is $\rho = 4$. Checking the endpoints will result in checking the following two series for convergence:

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)} \qquad\qquad \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

The sum on the left diverges (limit comparison with $\sum 1/n$), and the sum on the right converges conditionally using the Alternating Series Test (Be sure you can show both of these things).

That gives an interval of: $(-4, 4]$.

15. Straightforward- Very much like 10, 13 above.

22. In this case, we have an extra constant to keep track of, but we've had examples of it before. Let's try it generally- We'll assume that $b > 0$:

$$\sum_{n=2}^{\infty} \frac{b^n}{\ln(n)} (x-a)^n$$

Using the Ratio Test, we have:

$$\lim_{n\to\infty} \frac{b^{n+1}|x-a|^{n+1}}{\ln(n+1)} \cdot \frac{\ln(n)}{b^n |x-a|^n} = \lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} b|x-a| = b|x-a| < 1 \quad \Rightarrow \quad |x-a| < \frac{1}{b}$$

Therefore, the radius of convergence is $\rho = 1/b$. If we test the endpoints, $x = a - \frac{1}{b}$ and $x = a + \frac{1}{b}$, we get (respectively):

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}, \qquad \text{and} \qquad \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

As in #13, the sum on the left converges (using the Alternating Series Test), and the sum on the right diverges (limit comparison test with $\sum 1/n$). Therefore the interval of convergence is:

$$\left[a - \frac{1}{b}, a + \frac{1}{b}\right)$$

24. This one is included to give you practice dealing with the factorial. Notice that the quantity $2 \cdot 4 \cdot 6 = 2^3(1 \cdot 2 \cdot 3)$, or more generally,

$$2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n) = 2^n(1 \cdot 2 \cdot 3 \cdots n) = 2^n n!$$

This observation will make simplification much easier:

$$\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{n^2 x^n}{2^n n!}$$

Now perform the Ratio Test as usual, and you should find that the radius of convergence is infinity (that is, the series converges for every x). You get this when the limit in the Ratio Test is 0.

29. The key here is to infer from what is given that the power series $\sum_{n=1}^{\infty} c_n x^n$ converges when $x = 4$. Symmetry forces the interval of convergence to be at least $(-4, 4]$, so the series does not necessarily converge when $x = -4$, but definitely will converge (absolutely!) when $x = -2$.

Side Remark: We didn't have to assume that the series was centered at $x = 0$, although that made it easier to think about. For example, what if the series was centered at $x = -3$? Then we would say that

$$\sum_{n=0}^{\infty} c_n (x+3)^n$$

converges when $x + 3 = 4$, In this case, we now know that the interval of convergence must be at least $|x + 3| < 4$, which gives the interval $-7 < x < 1$. For part (a), we would set $x + 3 = -2$, or $x = -5$, which is inside our interval, and so again the series converges.

32. Fun to try it out- We have several examples from this homework. Don't fuss over it too much if you're having trouble- We're not looking for a specific answer here.

35. For this problem, only do part (a) using the Ratio Test.