# Homework Hints, 11.9 (Series as Geo Series)

1, 3, 7, 13(a,b), 40(a,b)

1. The radius of convergence for the derivative of a power series is the same as the radius of convergence for the series itself (Theorem 2, p. 748).

3. Use the Geo series with $r = -x$:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

and that converges if $|x| < 1$.

7. We want to manipulate the given expression until it resembles the sum of a geometric series. For example:

$$\frac{x}{9+x^2} = \frac{x}{9} \cdot \frac{1}{1+(x^2/9)} = \frac{x}{9} \cdot \frac{1}{1-(-x^2/9)}$$

We treat $x/9$ as a constant (with respect to $n$ and the ratio $r$), and the series this is a sum for is given by:

$$\sum_{n=0}^{\infty} \frac{x}{9} \cdot \left(\frac{-x^2}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{9^{n+1}}$$

The series converges when $x^2 < 9$, or $-3 < x < 3$. The series diverges for all other $x$.

13(a). The "trick" is to notice that

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{-1}{(1+x)^2}$$

Replacing the left side by its series (from #3) we have:

$$\frac{-1}{(1+x)^2} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n x^n\right) = \sum_{n=1}^{\infty} n(-1)^n x^{n-1}$$

Multiply by $(-1)$, and we get:

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} n(-1)^{n+1} x^{n-1}$$

and the series converges absolutely when $|x| < 1$. It is fine to leave your answer this way, but if you want to see how to make this the same as the back of the text, let $m = n - 1$. Then if $n = 1$, $m = 0$, and substitute into the sum. Notice that the sum should start with $+1$, then alternate.

$$\frac{1}{(1+x)^2} = \sum_{m=0}^{\infty} (m+1)(-1)^m x^m$$

13(b). Differentiate again:

$$\frac{-2}{(1+x)^3} = \sum_{m=1}^{\infty} m(m+1)(-1)^m x^{m-1}$$

Multiply both sides by $-1/2$ (the resulting sum begins with a positive term) and re-write the sum on the right by substituting $n = m - 1$ to get what's in the text:

$$\frac{1}{(1+x)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)(-1)^n x^n$$

40. These are good practice manipulating the geometric series.

(a) Find the sum of the series $\sum_{n=1}^{\infty} nx^{n_1}$

SOLUTION: Start with the standard geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and differentiate both sides with respect to $x$:

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, \qquad |x| < 1$$

(b) Similarly, for the first of the two sums, note that

$$\sum_{n=1}^{\infty} nx^n = \sum_{n=1}^{\infty} x \cdot nx^{n-1}$$

so multiply our previous expression by $x$:

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n, \qquad |x| < 1$$

For the second sum (in (ii)), set $x = 1/2$ from this sum so that

$$\frac{1/2}{(1-(1/2))^2} = \sum_{n=1}^{\infty} \frac{n}{2^n}$$

This expression simplifies to 2.