Section 8.2 HW Notes

5, 7, 9, 25

5. Given $y = x^3$, construct and simplify the integrand to get:

$$SA = \int_0^2 2\pi x^3 \sqrt{1+9x^4}\, dx$$

This is set up for u, du substitution, with $u = 1 + 9x^4$, $du = 36x^3\, dx$:

$$SA = \frac{2\pi}{36} \int_1^{145} u^{1/2}\, du = \frac{\pi}{27}(145^{3/2} - 1)$$

7. The integral in (7) becomes:

$$SA = \int_1^5 2\pi\sqrt{1+4x} \cdot \sqrt{\frac{5+4x}{1+4x}}\, dx = 2\pi \int_1^5 \sqrt{5+4x}\, dx$$

which we finish using $u = 5 + 4x$, $du = 4\, dx$.

9. With $y = \sin(\pi x)$, the arc length term becomes:

$$\sqrt{1 + \pi^2 \cos^2(\pi x)}$$

so the integral becomes

$$\int_0^1 2\pi \sin(\pi x)\sqrt{1+\pi^2\cos^2(\pi x)}\, dx$$

Let $u = \pi\cos(\pi x)$, so $du = -\pi^2 \sin(\pi x)$. Making the substitution:

$$SA = \frac{2}{\pi}\int_{-\pi}^{\pi} \sqrt{1+u^2}\, du = \frac{4}{\pi}\int_0^{\pi} \sqrt{1+u^2}\, du$$

Now with trig substitution, we have: $u = \tan(\theta)$. We could change the bounds, but they would become $\tan^{-1}(\pi)$ and $\tan^{-1}(-\pi)$, so let's wait on those. Continuing, we have: $du = \sec^2(\theta)\, d\theta$ so the integral becomes the following, found using the table we were provided.

$$\frac{4}{\pi}\int \sec(\theta)\sec^2(\theta)\, d\theta = \frac{2}{\pi}\left(\sec(\theta)\tan(\theta) + \ln|sec(\theta) + \tan(\theta)|\right)$$

Back substituting to u, we get $\sec(\theta) = \sqrt{1+u^2}$, so

$$\frac{2}{\pi}\left(u\sqrt{1+u^2} + \ln|\sqrt{1+u^2} + u|\right) = \frac{2}{\pi}(\pi\sqrt{1+\pi^2} + \ln|\sqrt{1+\pi^2} + \pi|$$

25. Gabriel's horn is a somewhat famous example in calculus of an object that has finite volume and infinite surface area!

(Not asked for, but we'll go ahead and compute it) If we compute the volume using disks, we have:

$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 \, dx$$

which we know converges from earlier. It's easy to compute it:

$$V = -\frac{\pi}{x}\Big|_1^\infty = 0 - -\pi = \pi$$

Now the surface area is computed as normal:

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}} \, dx = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3} \, dx \geq 2\pi \int_1^\infty \frac{\sqrt{x^4}}{x^3} \, dx = 2\pi \int_1^\infty \frac{1}{x} \, dx$$

We know this last integral diverges, therefore the surface area diverges.