Selected Solutions, Section 5,1

2. The purpose of this exercise is to be sure we have a little experience with estimating area using rectangles. The "midpoint" rule is to evaluate the heights of the rectangles at the midpoint of each rectangle.

   If we have heights $h_1, h_2, \ldots, h_n$, and the widths of the rectangles are all the same, Δx, then area is approximated by the sum of the areas of the rectangles (height $\times$ width):

   $$A = h_1\Delta x + h_2\Delta x + \cdots + h_n\Delta x = (h_1 + h_2 + \cdots + h_n)\Delta x$$

   (a) For this function, using 6 rectangles, the width of each rectangle is 2, and the 7 endpoints of the rectangles are:

   $$0, 2, 4, 6, 8, 10, 12$$

   Now, for L_6, evaluate the heights at the left endpoints: $0, 2, 4, 6, 8, 10$. We'll then sum the areas (the heights are estimated from the graph):

   $$2(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1) = 86.6$$

   For R_6, evaluate (estimate in this case) the heights at the right endpoints: $2, 4, 6, 8, 10, 12$:

   $$2(8.8 + 8.2 + 7.3 + 5.9 + 4.1 + 1) = 70.6$$

   For M_6, evaluate the heights at the midpoints, $1, 3, 5, 7, 9, 11$:

   $$2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8) = 79.4$$

   (b) For a decreasing function, left endpoints give an overestimate.

   (c) For a decreasing function, right endpoints give an underestmate.

   (d) In this case, the midpoints seem to give the best estimate.

13. In this situation, the velocity is decreasing (just as it did in (2.) Therefore (we have 7 points, so 6 rectangles), L_6 will be an overestimate, and R_6 will be an underestimate. You should find that these are approximately 34.7 and 44.8, resp.

17. Same idea, different application, as #2 and #17.

19. Let's get a formula for the area using a limit of n rectangles (with right endpoints).

    Since the interval is $1 \le x \le 3$, the width of each of the n rectangles will be

    $$\Delta x = \frac{3-1}{n} = \frac{2}{n}$$

    and the right endpoints are:

    $$1, \quad 1 + \frac{2}{n}, \quad 1 + \frac{4}{n}, \quad 1 + \frac{6}{n}, \cdots, \quad 1 + \frac{2(n-1)}{n}, \quad 3$$

so the "i^{th} right endpoint", $x_i = 1 + \frac{2}{n} i$. At this point, the height of the rectangle is given by

$$f(x_i) = \frac{2x_i}{x_i^2 + 1} = \frac{2(1 + 2i/n))}{(1 + 2i/n)^2 + 1}$$

so that the area under the curve is the limit, as the number of rectangles goes to infinity:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2(1 + 2i/n))}{(1 + 2i/n)^2 + 1} \cdot \frac{2}{n}$$