Calculus II

For the Exam...

- The exam will be about $1\frac{1}{2}$ times the length of a normal exam, and we have twice the amount of time to take it, so if you're well prepared, then time should not be an issue.
- As a reminder- If you do well on the final, then your lowest exam score will be replaced by the average of it and the final, so try your best!
- No calculators will be allowed, and no notes. I will provide the table of integrals (and sum formulas) that were given to you in class.
- If you are free during those times, you may switch sections for the final exam- Please let me know a day or two in advance so I know how many copies I'll need and where you'll be.

The Integral in Theory

- The definition of the definite integral.
  - Write an integral from a Riemann sum, and a Riemann sum from an integral.
- Interpret the integral in terms of geometry (area of a circle or triangle, for example)
- The Fundamental Theorem of Calculus, Part I.

  The primary condition is that the integrand, $f(x)$ is continuous on $[a, b]$. If

  $$g(x) = \int_a^x f(t)\,dt$$

  then g is continuous and differentiable, and $g'(x) = f(x)$.

  Corollary:

  $$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$$

  NOTE: The function g is a particular antiderivative- It is the antiderivative of f that so that $g(a) = 0$.
- The Fundamental Theorem of Calculus, Part II. The main computational tool of Calculus: If F is any antiderivative of the continuous function f,

  $$\int_a^b f(x)\,dx = F(b) - F(a)$$

- Understand the difference in notation: $\int f(x)\,dx \qquad \int_a^x f(t)\,dt \qquad \int_a^b f(x)\,dx$

- Understand the difference in notation: $\int_a^b \frac{d}{dx} f(x)\,dx \qquad \frac{d}{dx}\int_a^x f(t)\,dt \qquad \frac{d}{dx}\int_a^b f(x)\,dx$
- The Mean Value Theorem for Integrals. The average value of f is attained at some c in $[a, b]$. That is, if f is continuous on $[a, b]$, then there is a c in the interval so that:

$$f_{\text{avg}} = f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

- The improper integral (Types I and II) is approximated by a definite integral, and is defined by taking the limit. For example,

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

  NOTE: We need to recall techniques for computing a limit. For example, (i) algebraically simplify, (b) divide by x^n for some n, (c) l'Hospital's rule.

The Integral in Practice

We had several methods to evaluate an integral:

- Using geometry.
- u, du, or Substitution (Backwards Chain Rule)
- u, dv, or Integration by Parts (be able to use the tabular form of this)
- Partial Fractions. Also, be able to integrate something of the form $\int \frac{ax+b}{x^2+c}\,dx$
- Powers of sine and cosine. Remember the formulas for $\sin^2(x)$ and $\cos^2(x)$, and the main "trick" is to reserve something to get a substitution.
- Trigonometric substitution and the use of reference triangles.

  Primarily, understand what we can substitute in each case using a trig identity:

$$a^2 - u^2, \qquad u^2 + a^2 \qquad u^2 - a^2$$

  For example, in the first case, we substitute $u = a\sin(\theta)$, the expression simplifies to $a^2\cos^2(\theta)$ (a perfect square).
- The table of integrals can be used (See the handout) as well.

Applications of the Integral

- Be able to compute the volume of a solid of revolution using disks, washers and shells. Let w be either x or y, depending on how the functions are defined. Then:

$$\int_a^b \pi R^2\,dw \qquad \pi\int_a^b (R^2 - r^2)\,dw \qquad \int_a^b 2\pi r h\,dw$$

- Be able to compute the arc length of a curve.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy$$

  Then the arc length is $\int_a^b ds$

- Be able to compute simple *work* integrals. Constants for gravity and/or water would be provided.

Sequences to Series to Power Series to Taylor Series

Note the evolution of our notation in these sections:

$$\{a_n\}_{n=1}^{\infty}, \quad \sum_{k=1}^{\infty} a_k, \quad \sum_{k=1}^{\infty} c_k(x-a)^k, \quad \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Sequences:
  - What is a sequence?
  - Be able to determine if a sequence converges or diverges (Monotonic Sequence Theorem can be used, l'Hospital's rule, divide by an appropriate quantity, etc.)
- Series: $\sum_{n=1}^{\infty} a_n$
  - Template series: Geometric Series (and the formula for the sum of a geometric series), $p-$series, harmonic series, alternating harmonic series.
  - Convergence of the Series:
    * Test for divergence.
    * (For positive series) The direct ($a_n \leq b_n$) and limit comparison ($\lim_{n\to\infty} \frac{a_n}{b_n}$) tests.
    * (For positive series) The integral test, where $f(n) = a_n$- We integrate $f(x)$.

* (For abs convergence) The Ratio Test and Root Tests. The Ratio Test is by far the most widely used test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

* Check conditional convergence last: Alternating Series Test.
(The series has terms with alternating signs, the (abs value of the) terms are decreasing and the limit is zero).

- Power Series: $\sum_{k=1}^{\infty} c_k (x-a)^k$
  - We have one of three choices for convergence. The series converges: (i) Only at $x = a$, (ii) for all x, or (iii) for $|x - a| < R$, and diverges for $|x - a| > R$. We say that R is the radius of convergence.
  - Convergence is usually determined by the Ratio Test. We must check the endpoints of the interval separately (which gives the *interval of convergence*).
  - Be able to get new series from a given series by differentiation or integration.
- Taylor Series: $\sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ or Maclaurin: $\sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$
  - Construct a Taylor series for an *analytic* function f based at $x = a$ (or a Maclaurin series, which is a Taylor series based at $a = 0$).
  - Template series: e^x, $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$
  - Find the sum of a series by recognizing it as a familiar Taylor series.

Template Series

$$e^x, \quad \sin(x), \quad \cos(x), \quad \frac{1}{1-x}$$

Trig to know

$$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \tan^2(\theta) + 1 = \sec^2(\theta)$$

From this, we get the identities for $\tan(\theta)$ and $\sec(\theta)$ that are used in trig substitution. Also:

$$\sin(2x) = 2\sin(x)\cos(x) \qquad \cos(2x) = \cos^2(x) - \sin^2(x)$$

From the one on the right, we can get the half angle formulas needed for:

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \qquad \cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$