Exam 1 Review Solutions

1. State the Fundamental Theorem of Calculus:

   This is the summarized version on pg. 387, which is fine:

   Let f be continuous on $[a, b]$.

   - If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
   - $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f.

2. Give the *definition* of the definite integral (using rectangles of equal width and evaluating heights using right endpoints):

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right)\left(\frac{b-a}{n}\right)$$

3. True or False (and give a short reason):

   (a) $\int_0^2 (x - x^3)\,dx$ represents the area under the curve $y = x - x^3$ from 0 to 2.
   FALSE. The function is negative for $1 \leq x \leq 2$, so the integral represents the *net area* between the curve and the x axis. If you wanted the actual area, you would need to integrate $|x - x^3|$.

   (b) If $3 \leq f(x) \leq 5$ for all x, then $6 \leq \int_1^3 f(x)\,dx \leq 10$
   TRUE. We're using the property that, if $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

   (c) If f, g are continuous and $f(x) \geq g(x)$ for all $a < x < b$, then
   $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
   TRUE. This is a property of definite integrals.

   (d) If $f(x) \geq g(x)$ for all $a < x < b$, then $f'(x) \geq g'(x)$ for $a < x < b$.
   FALSE. For example, $f(x) = 10$ and $g(x) = 3x$ for $0 \leq x \leq 2$. Then $f'(x) = 0$, but $g'(x) = 3$.

   (e) All continuous functions have derivatives.
   FALSE. The famous example from Calc I is $y = |x|$ at $x = 1$. Continuity does not imply differentiability.

   (f) All continuous functions have antiderivatives.
   TRUE. This is what the Fundamental Theorem of Calculus says- $g(x) = \int_a^x f(t)\,dt$ is an antiderivative if f is continuous.

   (g) If f has a discontinuity at $x = 0$, then $\int_{-1}^{1} f(x)\,dx$ does not exist.

FALSE. If f has a discontinuity at $x = 0$, then the FTC does not apply in any interval containing zero. But consider the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x > 0 \end{cases}$$

This function is not continuous at the origin, but the **area** described by $\int_{-1}^{1} f(x)\,dx$ would be the sum of the two rectangles, or 3.

4. The speed of a runner increased steadily during the first part of a race. Her speed (in feet per sec) at half-second intervals is given in the table. Give an upper and lower estimate for the distance she has run during these two seconds:

t	0	0.5	1.0	1.5	2.0
$v(t)$	0	3	6	7	10

SOLUTION: Since the function $v(t)$ is increasing, then using right endpoints will result in an overestimate, and left endpoints will be an underestimate:

An underestimate:

$$0 \cdot 0.5 + 3 \cdot 0.5 + 6 \cdot 0.5 + 7 \cdot 0.5 = \frac{1}{2}(0 + 3 + 6 + 7) = \frac{16}{2} = 8$$

An overestimate (and factoring out the width of 1/2):

$$\frac{1}{2}(3 + 6 + 7 + 10) = \frac{26}{3} = 12$$

5. Evaluate the integral, if it exists

(a) $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du$

SOLUTION: Use algebra first to simplify.

$$\int_1^9 u^{-1/2} - 2u\,du = 2u^{1/2} - u^2\Big|_1^9 = (2(3) - 81) - (2 - 1) = -76$$

(b) $\displaystyle\int 3^x + \frac{1}{x} + \sec^2(x)\,dx$

SOLUTION: These are an assortment of functions from the table:

$$\frac{1}{\ln(3)}3^x + \ln|x| + \tan(x) + C$$

(c) $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4\tan(t)}{2 + \cos(t)}\,dt$

SOLUTION: By symmetry (the function is odd, since $\tan(t)$ is odd, and is multiplied by an even function), the integral is zero.

(d) $\displaystyle\int_0^3 |x^2 - 4|\, dx$
SOLUTION: Break up the interval to get rid of the absolute value:

$$\int_0^2 -(x^2 - 4)\, dx + \int_2^3 x^2 - 4\, dx = \cdots = \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$$

(e) $\displaystyle\int \frac{\cos(\ln(x))}{x}\, dx$
SOLUTION: Use $u = \ln(x)$, $du = 1/x\, dx$ so:

$$\int \frac{\cos(\ln(x))}{x}\, dx = \int \cos(u)\, du = \sin(u) + C = \sin(\ln(x)) + C$$

(f) $\displaystyle\int_0^2 \sqrt{4 - x^2}\, dx$
SOLUTION: Use geometry to see this is the area of a quarter circle-

$$y = \sqrt{4 - x^2} \quad \Rightarrow \quad x^2 + y^2 = 4$$

so the area is π.

(g) $\displaystyle\int \frac{1}{\sqrt{1 - x^2}}\, dx$
SOLUTION: The integrand is the derivative of $\sin^{-1}(x)$, so the answer is $\sin^{-1}(x) + C$.

(h) $\displaystyle\int_{-2}^2 \frac{1}{x}\, dx$
SOLUTION: This is a bit of a trick question, since it looks like symmetry is at work. However, the function is not continuous on the interval $[-2, 2]$, so we would say that the FTC does not apply. We'll have a technique in a later chapter that will allow us to work with these kinds of functions.

(i) $\displaystyle\int_0^1 (\sqrt[4]{w} + 1)^2\, dw$
SOLUTION: Multiply it out first: $(w^{1/4} + 1)^2 = w^{1/2} + 2w^{1/4} + 1$, so

$$\int w^{1/2} + 2w^{1/4} + 1\, dw = \frac{2}{3}w^{3/2} + \frac{8}{5}x^{5/4} + w + C$$

(j) $\displaystyle\int_{-2}^{-1} \frac{1}{x}\, dx = \ln|x|\Big|_{-2}^{-1} = \ln(1) - \ln(2) = 0 - \ln(2) = \ln(1/2)$

(k) $\displaystyle\int_0^{1/2} \frac{\sin^{-1}(x)}{\sqrt{1 - x^2}}\, dt$
SOLUTION: Let $u = \sin^{-1}(x)$, so $du = dx/\sqrt{1 - x^2}$. Substitute, with $\sin^{-1}(0) = 0$ and $\sin^{-1}(1/2) = \pi/6$, since $\sin(\pi/6) = 1/2$:

$$\int_0^{\pi/6} u\, du = \frac{1}{2}u^2\Big|_0^{\pi/6} = \frac{\pi^2}{72}$$

(l) $\int (1+\tan(t))\sec^2(t)\,dt$

SOLUTION: Let $u = \tan(t)$, so $du = \sec^2(t)\,dt$, and the integral becomes

$$\int 1+u\,du = u + \frac{1}{2}u^2 = \tan(t) + \frac{1}{2}\tan^2(t) + C$$

NOTE: Letting $u = 1+\tan(t)$ would also be OK. In that case, you'll get something that may look a little different:

$$\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(1+\tan(t))^2 + C = \frac{1}{2} + \tan(t) + \frac{1}{2}\tan^2(t) + C$$

(m) $\int \tan(x)\,dx$

SOLUTION: Re-write the integrand as $\sin(x)/\cos(x)$, then let $u = \cos(x)$. Therefore,

$$\int \frac{\sin(x)}{\cos(x)}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$$

(Its OK if you don't do the last step).

(n) $\int x\sqrt{1+x}\,dx$

SOLUTION: Let $u = 1+x$. Then $du = dx$ and if we substitute now, we see there is an extra x. Go to the first equation, $u = 1+x$ and solve for x in terms of u: $x = 1-u$, and now we have:

$$\int (1-u)u^{1/2}\,du = \int u^{1/2} - u^{3/2}\,du = \frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2} + C = \frac{2}{3}(1+x)^{3/2} - \frac{2}{5}(1+x)^{5/2} + C$$

(o) $\int \frac{y-1}{\sqrt{3y^2-6y+4}}\,dy$

SOLUTION: Let $u = 3y^2 - 6y + 4$, so $du = 6y - 6\,dy = 6(y-1)\,dy$. Substitute into the equation and:

$$\frac{1}{6}\int u^{-1/2}\,du = \frac{1}{3}u^{1/2} + C = \frac{1}{3}(3y^2-6y+4)^{1/2} + C$$

(p) $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

SOLUTION: Let $u = 1/x$, so that $du = -1/x^2\,dx$. Further, if $x = 1$ then $u = 1$, and if $x = 2$, then $u = 1/2$:

$$-\int_1^{1/2} e^u\,du = \int_{1/2}^1 e^u\,du = e^u\Big|_{1/2}^1 = e - \sqrt{e}$$

(q) $\int_1^2 x\sqrt{x-1}\,dx$

SOLUTION: This is very similar to a previous problem, but this one is a definite integral. Let $u = x-1$ so that $du = dx$. For the bounds, u will run from 0 to 1.

$$\int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 u^{3/2} + u^{1/2}\,du = \left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right|_0^1 = \frac{16}{15}$$

(r) $\int_0^1 \frac{1}{(1+\sqrt{x})^4}\,dx$

SOLUTION: The substitution is a little tricky. Let $u = 1+\sqrt{x}$, as we might naturally choose. Then

$$du = \frac{1}{2\sqrt{x}}\,dx \quad \Rightarrow \quad 2\sqrt{x}\,du = dx$$

Writing the left side of this expression all in terms of u, we have:

$$2(u-1)\,du = dx$$

and now we can proceed as usual:

$$2\int_1^2 (u-1)u^{-4}\,du = 2\int_1^2 u^{-3} - u^{-4}\,du = 2\left(-\frac{1}{2}u^{-2} + \frac{1}{3}u^{-3}\right|_1^2 =$$

$$2\left[\left(-\frac{1}{8} + \frac{1}{24}\right) - \left(-\frac{1}{2} + \frac{1}{3}\right)\right] = \frac{1}{6}$$

6. Find the derivative of the function:

   Note: For each of these, we're using the formula from the FTC part I

$$y = \int_{g(x)}^{h(x)} f(t)\,dt \quad \Rightarrow \quad y' = f(h(x))h'(x) - f(g(x))g'(x)$$

   (a) $F(x) = \int_0^{x^2} \frac{\sqrt{t}}{1+t^2}\,dt$ so $F'(x) = \frac{2x^2}{1+x^2}$, assuming $x > 0$.

   (b) $y = \int_{\sqrt{x}}^{3x} \frac{e^t}{t}\,dt$

$$y' = \frac{3e^{3x}}{3x} - \frac{1}{2\sqrt{x}}\frac{e^{\sqrt{x}}}{\sqrt{x}} = \frac{e^{3x}}{x} - \frac{e^{\sqrt{x}}}{2x}$$

7. The integrals below could have different ranges (and therefore also use different integrands), but I think these are probably the most straightforward:

$$\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots \left(\frac{n}{n}\right)^9\right] = \int_0^1 x^9\,dx = \frac{1}{10}x^{10}\Big|_0^1 = \frac{1}{10}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(5 + \frac{2i}{n}\right)^4 = \int_5^7 x^4\,dx = \frac{1}{5}(7^5 - 5^5) = \frac{13682}{5}$$

   (Sorry about the large numbers).

8. Evaluate the following definite integral via the definition (that is, using the Riemann sum).

$$\int_1^4 x^2 - 4x + 2\,dx$$

SOLUTION: The widths of the rectangles will be $3/n$, and the "i^{th} right endpoint is given by $1 + \frac{3i}{n}$. Now the Riemann sum will be:

$$\int_1^4 x^2 - 4x + 2\,dx = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\left(1 + \frac{3i}{n}\right)^2 - 4\left(1 + \frac{3i}{n}\right) + 2 \right)$$

First simplify what's inside the parentheses:

$$\left(\left(1 + \frac{3i}{n}\right)^2 - 4\left(1 + \frac{3i}{n}\right) + 2 \right) = 1 + \frac{6i}{n} + \frac{9i^2}{n^2} - 4 - \frac{12i}{n} + 2 = -1 - \frac{6i}{n} + \frac{9i^2}{n^2}$$

Go back to the sum:

$$\sum_{i=1}^{n} \left(\left(1 + \frac{3i}{n}\right)^2 - 4\left(1 + \frac{3i}{n}\right) + 2 \right) = -\sum_{i=1}^{n} 1 - \frac{6}{n} \sum_{i}^{n} i + \frac{9}{n^2} \sum_{i=1}^{n} i^2 =$$

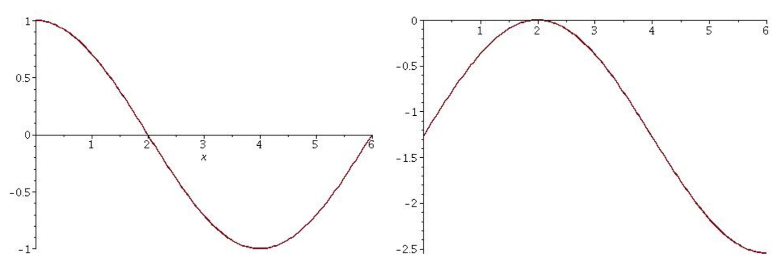
$$-n - 3(n+1) + \frac{3}{2} \frac{(n+1)(2n+1)}{n}$$

Multiply by $3/n$, then take the limit as $n \to \infty$:

$$\lim_{n\to\infty} -3 - 9\frac{n+1}{n} + \frac{9}{2} \frac{(n+1)(2n+1)}{n^2} = -3 - 9 + 9 = -3$$

9. If $F(x) = \int_2^x f(t)\,dt$, where f is given below, which of the following is the smallest, and which is the largest?

(a) $F(0)$ (b) $F(1)$ (c) $F(2)$ (d) $F(3)$ (e) $F(4)$



SOLUTION: The graph of the antiderivative is included above for your inspection, but we can answer the question piecemeal as below:

To start, let's estimate some of these areas. $F(0)$ will be a negative area- about half the value of the integral from 2 to 6.

The value of $F(1)$ will still be negative, but closer to zero. The value of $F(2)$ is zero, then the area becomes more and more negative as x increases. $F(3)$ is about the same as $F(1)$, so there's a tie for most negative- $F(0)$ and $F(4)$.

The largest value of F is at $F(2)$.

10. Continuing with the function F in Problem 9, at what values of x does F have local maximum and minimum values? On what interval(s) is F concave downward?

SOLUTION: F will have a local maximum when F' changes from positive to negative. That occurs at $t = 2$. F would have a local minimum at $t = 6$ if the curve were to continue through the end of the graph.

If $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ and $F''(x) = f'(x)$. Therefore, the curve F is concave down when $f'(x)$ is negative. This is when f is decreasing (between times 0 and 4). Similarly, F is concave up when $f'(x)$ is positive, which in this graph is for times from 4 to 6.

11. Prove the following using Riemann sums (for credit, you must use the Riemann sum):

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2}$$

SOLUTION: Take the limit of the Riemann sum (right endpoints):

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} = \lim_{n\to\infty} \frac{b-a}{n}\left[\sum_{i=1}^{n} a + \frac{b-a}{n}\sum_{i=1}^{n} i\right] =$$

$$\lim_{n\to\infty} \frac{b-a}{n}\left[an + \frac{b-a}{n}\cdot\frac{n(n+1)}{2}\right] =$$

$$\lim_{n\to\infty} (b-a)a + \frac{(b-a)^2}{2}\cdot\frac{n+1}{n} = ab - b^2 + \frac{1}{2}(b^2 - 2ab + a^2) = \frac{b^2 - a^2}{2}$$

12. Evaluate (Solutions given below):

(a) $\displaystyle\int_0^1 \frac{d}{dx}\left(e^{\tan^{-1}(x)}\right)dx = e^{\tan^{-1}(1)} - e^{\tan^{-1}(0)} = e^{\pi/4} - 1$

Generally speaking, this is $\int_a^b f'(x)\,dx = f(b) - f(a)$.

(b) $\displaystyle\frac{d}{dx}\int_0^1 e^{\tan^{-1}(x)}\,dx = 0$ (This is the derivative of a constant).

(c) $\displaystyle\frac{d}{dx}\int_0^x e^{\tan^{-1}(t)}\,dt = e^{\tan^{-1}(x)}$. This is FTC, part 1.

13. If $f(x) = \displaystyle\int_0^{\sin(x)} \sqrt{1+t^2}\,dt$ and $g(y) = \displaystyle\int_3^y f(x)\,dx$, then find $g''(\pi/6)$.

SOLUTION: First, $g'(y)$ is found by applying the FTC, part 1:

$$g'(y) = f(y) = \int_0^{\sin(y)} \sqrt{1+t^2}\,dt$$

so that $g''(y) = \sqrt{1+\sin^2(y)}\,\cos(y)$. Now, $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$, so the overall answer is:

$$g''(\pi/6) = \sqrt{1+\frac{1}{4}}\,\frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{4}$$

14. A particle moves along a line with velocity $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 5]$.

SOLUTION: The displacement will simply be the integral:

$$\int_0^5 t^2 - t\, dt = \left(\frac{1}{3}t^3 - \frac{1}{2}t^2\right|_0^5 = \frac{175}{6} \approx 29.17$$

Sorry about the fractions- I'll try to keep the numbers somewhat nice for the exam.

The distance traveled is the absolute value of the velocity. Notice that the velocity function is an upward opening parabola with zeros at 0 and 1. Therefore,

$$\int_0^5 |t^2 - t|\, dt = \int_0^1 -t^2 + t\, dt + \int_1^5 t^2 - t\, dt = \frac{1}{6} + \frac{88}{3} = \frac{59}{2} = 29.5$$

15. A bacteria population is 2500 at time $t = 0$ and its rate of growth is $1000 \cdot 2^t$ bacteria per hour after t hours. What is the population after 1 hour?

SOLUTION: If $P(t)$ is the population at time t, then $P'(t) = 1000 \cdot 2^t$, and the net change in population after 1 hour is given by:

$$P(1) - P(0) = \int_0^1 P'(t)\, dt$$

Since we want the actual population and not the net population, we'll add in the population at time $t = 0$. That is, the actual population is:

$$P(0) + (P(1) - P(0)) = P(0) + \int_0^1 1000 \cdot 2^t\, dt = 2500 + 1000\left(\frac{1}{\ln(2)}2^t\right|_0^1 = 2500 + \frac{1000}{\ln(2)}$$

which is approximately 3943.

16. If f is continuous and $\int_0^9 f(x)\, dx = 4$, find $\int_0^3 xf(x^2)\, dx$

SOLUTION: Use u, du substitution with $u = x^2$, so $du = 2x\, dx$, and

$$\int_0^3 xf(x^2)\, dx = \frac{1}{2}\int_0^9 f(u)\, du = \frac{1}{2} \cdot 4 = 2$$

17. If $f''(x) = 2 - 12x$, $f(0) = 0$ and $f(2) = 15$, find $f(x)$.

SOLUTION: $f'(x) = 2x - 6x^2 + C_1$ so $f(x) = x^2 - 2x^3 + C_1x + C_2$. We use the information provided to solve for the constants:

$$f(0) = 0 \quad \Rightarrow \quad 0 + 0 + 0C_1 + C_2 = 0$$

so

$$f(2) = 15 \quad \Rightarrow 2^2 - 22^3 + 2C_1 = 15 \quad C_1 = \frac{27}{2}$$

Therefore

$$f(x) = x^2 - 2x^3 + \frac{27}{2}x$$

18. Evaluate the following integral using geometry (and area). You might use a property of integrals first.

$$\int_0^1 x + \sqrt{1-x^2}\,dx$$

SOLUTION: First,

$$\int_0^1 x + \sqrt{1-x^2}\,dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1-x^2}\,dx$$

The first quantity is a triangle with base and height 1 (so that area is 1/2). The second quantity represents a quarter circle of radius 1, so that area is $\pi/4$. Sum them to get the overall area:

$$\frac{1}{2} + \frac{\pi}{4}$$

19. Given the following sum, first express it using sigma notation, then find the limit.

$$\lim_{n\to\infty} \frac{3}{n}\left[\left(\frac{1}{n^2}+1\right) + \left(\frac{4}{n^2}+1\right) + \left(\frac{9}{n^2}+1\right) + \cdots \left(\frac{n^2}{n^2}+1\right)\right]$$

SOLUTION: As a sum, the expression above is given by the following, which we then begin to evaluate:

$$\lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left(\frac{i^2}{n^2}+1\right) = \lim_{n\to\infty} \frac{3}{n}\left(\frac{1}{n^2}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 1\right) = \lim_{n\to\infty}\left(\frac{3}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}n\right) = 4$$

Side Remark: It is possible to write this as a Riemann sum, but it may not be obvious. This expression could be the Riemann sum for $\int_0^3 (x/3)^2 + 1\,dx$

20. Find four different integrals that are represented by the following Riemann sum: $\sum_{i=1}^{n}\left(\frac{3}{n}\right)\left[4 - \frac{9i^2}{n^2}\right]$

SOLUTION: The wide of the interval is 3 units. We also have:

$$f\left(a + \frac{3i}{n}\right) = 4 - \frac{9i^2}{n}$$

Therefore, one way to get the right f is to take $a + \frac{3i}{n}$, subtract a, then take 4 minus the square of the result. That is,

$$4 - \frac{9i^2}{n} = 4 - \left[\left(a + \frac{3i}{n}\right) - a\right]^2$$

Here are some choices:

$$\int_0^3 4 - x^2\,dx \qquad \int_1^4 4 - (x-1)^2\,dx \qquad \int_2^5 4 - (x-2)^2\,dx \qquad \int_{-1}^2 4 - (x+1)^2\,dx$$