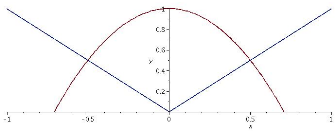
Solutions to the Review Questions, Exam 2

1. Find the area bounded between the regions $y = 1 - 2x^2$ and $y = |x|$.

   SOLUTION: Looking at the graph below, the area (using symmetry) can be written as:

   $$2\int_0^{1/2} (1-2x^2) - x\,dx = 2\left(x - \frac{1}{2}x^2 - \frac{2}{3}x^3\right|_0^{1/2} = \frac{12-3-2}{12} = \frac{7}{12}$$

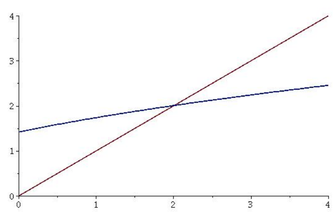
2. Evaluate the integral and interpret it as the area of a region (sketch it).

   $$\int_0^4 |\sqrt{x+2} - x|\,dx$$

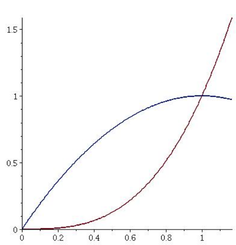
   SOLUTION: The is the area between $y = \sqrt{x+2}$ and $y = x$ (the functions switch at $x = 2$), as shown below. The area is:

   $$\int_0^2 \sqrt{x+2} - x\,dx + \int_2^4 x - \sqrt{x+2}\,dx = \left(\frac{2}{3}(x+2)^{3/2} - \frac{1}{2}x^2\right|_0^2 + \left(\frac{1}{2}x^2 - \frac{2}{3}(x+2)^{3/2}\right|_2^4$$

   Numerically, this is approximately 2.983.

3. Let R be the region in the first quadrant bounded by $y = x^3$ and $y = 2x - x^2$. Calculate the following quantities: (Exam note: Region R would typically be plotted for you).

   (a) The area of R:

   $$\int_0^1 2x - x^2 - x^3\,dx = \left(x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right|_0^1$$

   $$1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$$

   (b) Volume obtained by rotating R about the $x-$axis.

   SOLUTION: Using washers, the rectangle is a strip with width Δx and height going from x^3 to $2x - x^2$. Therefore,

   $$V = \int_0^1 \pi(R^2 - r^2)\,dx = \int_0^1 \pi((2x-x^2)^2 - x^6)\,dx = \pi\int_0^1 -x^6 + x^4 - 4x^3 + 4x^2\,dx$$

Evaluating, we get 41/105. Note that we would not want to use washers in this case, since then we would be integrating in y, and you would need to invert $y = 2x - x^2$ (which could be done, but does take a little work).

(c) Volume obtained by rotating R about the $y-$axis.

SOLUTION: Use shells in this case so that we keep the integral in terms of x. The height of the shell is $(2x - x^2) - x^3$, the radius of the shell is x, and the width is Δx. Therefore, the full volume is

$$V = \int_0^1 2\pi rh\,dx = \int_0^1 2\pi x(2x - x^2 - x^3)\,dx = \frac{13\pi}{30}$$

4. Use any method to find an integral representing the volume generated by rotating the given region about the specified axis. You do NOT need to evaluate the integral:

(a) $y = \sqrt{x}, y = 0, x = 1$; about $x = 2$.

(b) $y = x^2$, $y = 2 - x^2$; about $x = 1$.

(c) $y = x^2$, $y = 2 - x^2$; about $y = -3$.

(d) $y = \tan(x)$, $y = x$, $x = \pi/3$; about the $y-$axis.

5. Write the partial fraction decomposition for each of the following (do not actually solve for the coefficients):

(a) $\dfrac{3 - 4x^2}{(2x+1)^3} = \dfrac{A}{2x+1} + \dfrac{B}{(2x+1)^2} + \dfrac{C}{(2x+1)^3}$

(b) $\dfrac{7x - 41}{(x-1)^2(2-x)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{2-x}$

(c) $\dfrac{x+1}{x^3(x^2 - x + 10)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx + E}{x^2 - x + 10} + \dfrac{Fx + G}{(x^2 - x + 10)^2}$

We note that $x^2 - x + 10$ is irreducible, since $b^2 - 4ac = 1 - 4 \cdot 10 < 0$.

6. Integrate the following:

$$\int \frac{2x^3 - x^2 - 4x - 13}{x^2 - x - 2}\,dx$$

SOLUTION: Do long division first:

$$\frac{2x^3 - x^2 - 4x - 13}{x^2 - x - 2} = 2x + 1 + \frac{x - 11}{x^2 - x - 2} = 2x + 1 + \frac{x - 11}{(x+1)(x-2)}$$

Now expand the last term:

$$\frac{x - 11}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

Solve for A, B: $x - 11 = A(x-2) + B(x+1)$. If we substitute $x = -1$, we get $-12 = -3A$, or $A = 4$. If we substitute $x = 2$, we get $-9 = 3B$, or $B = -3$. Therefore,

$$\frac{2x^3 - x^2 - 4x - 13}{x^2 - x - 2} = 2x + 1 + 4\,\frac{1}{x+1} - 3\,\frac{1}{x-2}$$

and the integral is

$$x^2 + x + 4\ln|x+1| - 3\ln|x-2| + C$$

7. If $x = \tan(\theta)$, show that $\sin(2\theta) = \frac{2x}{1+x^2}$.

   We run into something similar to this when we integrate using a trig substitution. In this case, use a reference triangle for the tangent, and note that

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2 \cdot \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} = \frac{2x}{1+x^2}$$

8. True or False? (And give a short reason)

   (a) To find $\int \sin^2(x)\cos^5(x)\,dx$, rewrite the integrand as $\sin^2(x)(1-\sin^2(x))^2\cos(x)$

   SOLUTION: That is true; then let $u = \sin(x)$ and $du = \cos(x)\,dx$.

   (b) Integration by parts is the integral version of the Product Rule for derivatives.

   SOLUTION: That is true. We showed it in class, but you could also start with the product rule, then integrate both sides:

$$(fg)' = f'g + fg' \quad \rightarrow \quad f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

   (c) To find $\int \frac{2x-3}{x^2-3x+5}\,dx$, start by completing the square in the denominator.

   SOLUTION: False- It's probably best to see if the denominator factors. If not, then look for an obvious u, du substitution- In this case, $u = x^2 - 3x + 5$ and $du = 2x - 3\,dx$

   (d) To find $\int \frac{3}{x^2-3x+5}\,dx$, start by completing the square in the denominator.

   SOLUTION: False. Start by checking that you cannot factor the denominator- In this case, we cannot, so then continue by completing the square.

   (e) To find $\int \frac{3}{x^2-4x+3}\,dx$, start by completing the square in the denominator.

   SOLUTION: False. Start by checking the denominator- In this case, we can factor it, so we should do that and use partial fractions.

   (f) u, du substitution is the integral version of the Chain Rule.

   SOLUTION: True. We showed it in class, and gives you some good insight into when to use it.

9. Evaluate using any method, unless specified below:

(a)

$$\int \frac{4\,dx}{(4+x^2)^{3/2}}$$

SOLUTION: Trig substitution is the most direct choice.
Let $x = 2\tan(\theta)$. Then

$$4 + x^2 = 4 + 4\tan^2(\theta) = 4\sec^2(\theta) \qquad \text{and} \qquad dx = 2\sec^2(\theta)\,d\theta$$

Substituting these in, we get:

$$\int \frac{8\sec^2(\theta)\,d\theta}{8\sec^3(\theta)} = \int \cos(\theta)\,d\theta = \sin(\theta) + C$$

Use the reference triangle to convert this answer back to x:

$$\frac{x}{\sqrt{4+x^2}} + C$$

(b) $\int \tan^3(x)\sec^2(x)\,dx$

SOLUTION: This is a trig integral- Try to reserve something to pull off a u, du substitution. In this case, reserve $\sec^2(x)$ so that $u = \tan(x)$ and $du = \sec^2(x)\,dx$, and the integral becomes $\int u^3\,du$.

$$\frac{1}{4}\tan^4(x) + C$$

(c) $\displaystyle\int \frac{3x+2}{x^2+6x+8}\,dx = \int \frac{3x+2}{(x+2)(x+4)}\,dx$

SOLUTION: Since the denominator factors, use partial fractions. Here is the final answer:

$$= \int \frac{5}{x+4} - \frac{2}{x+2}\,dx = 5\ln|x+4| - 2\ln|x+2| + C$$

(d) $\displaystyle\int \frac{t^2\cos(t^3-2)}{\sin^2(t^3-2)}\,dt$

SOLUTION: Look for the u, du substitution first. In this case, we do have what we need, if we let $u = \sin(t^3-2)$. Then the integral becomes

$$\frac{1}{3}\int u^{-2}\,du = -\frac{1}{3}\csc(t^3-2) + C$$

(e) $\int \cos^5(x)\sqrt{\sin(x)}\,dx$

SOLUTION: Look for a substitution first. Looks like we can reserve one of the cosines for the du term, and make $u = \sin(x)$:

$$\int \cos^4(x)\sqrt{\sin(x)}\,[\cos(x)\,dx] = \int (1-\sin^2(x))^2\sqrt{\sin(x)}\,[\cos(x)\,dx] =$$

$$\int (1-u^2)^2\sqrt{u}\,du = \int u^{1/2} - 2u^{5/2} + u^{9/2}\,du = \frac{2}{3}u^{3/2} - \frac{4}{7}u^{7/2} + \frac{2}{11}u^{11/2}$$

To finish up the problem, back substitute the x.

(f) $\displaystyle\int \frac{x}{x^2+4}\,dx$

SOLUTION: Straight u, du substitution: $\frac{1}{2}\ln|x^2+4| + C$.

(g) $\displaystyle\int \frac{dx}{\sqrt{1-6x-x^2}}$

SOLUTION: We'll need to complete the square in the denominator, then probably do a trig substitution. To complete the square, notice that

$$1 - 6x - x^2 = 1 - (x^2 + 6x + \quad) = 10 - (x+3)^2 = \sqrt{10}^2 - (x+3)^2$$

I can make the substitution: $x+3 = \sqrt{10}\sin(\theta)$ so that the denominator becomes:

$$\sqrt{10 - 10\sin^2(\theta)} = \sqrt{10}\cos(\theta)$$

and don't forget the dx term: $dx = \sqrt{10}\cos(\theta)\,d\theta$:

$$\int \frac{dx}{\sqrt{1-6x-x^2}} = \int \frac{\sqrt{10}\cos(\theta)\,d\theta}{\sqrt{10}\cos(\theta)} = \theta + C$$

Convert back to x to get

$$\sin^{-1}\left(\frac{x+3}{\sqrt{10}}\right) + C$$

(h) $\displaystyle\int \frac{x-1}{x^2+3}\,dx$

SOLUTION: It might be easiest to separate these into two integrals, or you could do a trig substitution. Separating we get:

$$\int \frac{x-1}{x^2+3}\,dx = \int \frac{x}{x^2+3}\,dx - \int \frac{1}{x^2+3}\,dx$$

The first integral is set up for u, du substitution. For the second integral, factor 3 from the denominator so that we can do a different u, du substitution:

$$\int \frac{1}{x^2+3}\,dx = \frac{1}{3}\int \frac{dx}{\left(\frac{x}{\sqrt{3}}\right)^2 + 1} = \frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}\,du = \frac{1}{\sqrt{3}}\tan^{-1}(u)$$

Put the two together: $\frac{1}{2}\ln|x^2+3| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

(i) $\int \sin^2(3t)\,dt$

SOLUTION: Use the half angle identity:

$$\int \sin^2(3t)\,dt = \frac{1}{2}\int 1-\cos(6t)\,dt = \frac{1}{2}t - \frac{1}{12}\sin(6t) + C$$

(j) $\int \frac{3x-2}{(x^2+2)^2}\,dx$

SOLUTION: We could break this into two, then use u, du substitution on one and trig substitution on the other, or we can just go for the trig substitution gusto from the start!

Let $x = \sqrt{2}\tan(\theta)$ and make the necessary substitutions to get:

$$\int \frac{3x-2}{(x^2+2)^2} = \int \frac{(3\sqrt{2}\tan(\theta)-2)(\sqrt{2}\sec^2(\theta))}{4\sec^4(\theta)}\,d\theta = \frac{\sqrt{2}}{4}\int \frac{(3\sqrt{2}\tan(\theta)-2)}{\sec^2(\theta)}d\theta$$

Continuing to simplify,

$$\frac{3}{2}\int \sin(\theta)\cos(\theta)\,d\theta - \frac{\sqrt{2}}{2}\int \cos^2(\theta)\,d\theta = \frac{3}{2}\int \sin(\theta)\cos(\theta)\,d\theta - \frac{\sqrt{2}}{4}\int (1+\cos(2\theta))\,d\theta$$

These can now each be evaluated to get:

$$\frac{3}{4}\sin^2(\theta) - \frac{\sqrt{2}}{4}\theta - \frac{\sqrt{2}}{8}\sin(2\theta) = \frac{3}{4}\sin^2(\theta) - \frac{\sqrt{2}}{4}\theta - \frac{\sqrt{2}}{4}\sin(\theta)\cos(\theta)$$

Finally, back substitute x using a triangle (which is why we converted $\sin(2\theta)$ in the previous answer). Unsimplified, the answer is:

$$\frac{3}{4}\left(\frac{x}{\sqrt{x^2+2}}\right)^2 - \frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{\sqrt{2}}{4}\frac{x}{\sqrt{x^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2+2}} + C$$

NOTE: If you evaluate $\int \sin(\theta)\cos(\theta)\,d\theta = -\frac{1}{2}\cos^2(\theta)$, you get a slightly different answer...

(k) $\int \sin^{-1}(x)\,dx$

Use integration by parts

$$\begin{array}{ccc} + & \sin^{-1}(x) & 1 \\ - & \frac{1}{\sqrt{1-x^2}} & x \end{array} \quad \Rightarrow \quad x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $u = 1-x^2$, $du = -2x\,dx$ to finish: $x\sin^{-1}(x) + \sqrt{1-x^2} + C$

(l) $\int x^3\sqrt{x^2+4}\,dx$

Substitution: $u = x^2 + 4$, $du = 2x\,dx$, and $x^2 = u - 4$. Then

$$\int x^3\sqrt{x^2+4}\,dx = \frac{1}{2}\int (u-4)u^{1/2}\,du = \frac{1}{2}\int u^{3/2} - 4u^{1/2}\,du$$

(and continue...)

$$\frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C$$

(m) $\int \sqrt{2x - x^2}\,dx$

Complete the square first: $\int \sqrt{-(x^2-2x+1)+1}\,dx = \int \sqrt{1-(x-1)^2}\,dx$ Use a trig substitution: $\sin(\theta) = x - 1$ and $\cos(\theta)\,d\theta = dx$. The integral becomes the following, which we can evaluate using either the half angle formulas or your table of formulas:

$$\int \cos^2(\theta)\,d\theta = \frac{1}{2}\cos(\theta)\sin(\theta) + \frac{1}{2}\theta$$

Use the reference triangle to convert back to x:

$$\frac{1}{2}(\sin^{-1}(x-1) + (x-1)\sqrt{2x-x^2}) + C$$

(n) $\int \sqrt{t}\ln(t)\,dt$

Integration by parts:

$$\begin{array}{ccc} + & \ln(t) & \sqrt{t} \\ - & 1/t & \frac{2}{3}t^{3/2} \end{array} \quad \Rightarrow \quad \frac{2}{3}t^{3/2}\ln(t) - \frac{2}{3}\int t^{1/2}\,dt = \frac{2}{3}t^{3/2}\ln(t) - \frac{4}{9}t^{3/2} + C$$

(o) $\int \frac{3x-1}{(x+2)(x-3)}\,dx$

By partial fractions,

$$\frac{3x-1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3} \quad \Rightarrow \quad 3x - 1 = A(x-3) + B(x+2)$$

Substitute $x = 3$ to get $A = 7/5$ and substitute $x = -2$ to get $B = 8/5$. Then the integral becomes:

$$\int \frac{3x-1}{(x+2)(x-3)}\,dx = \frac{7}{5}\int \frac{1}{x+2}\,dx + \frac{8}{5}\int \frac{1}{x-3}\,dx = \frac{7}{5}\ln|x+2| + \frac{8}{5}\ln|x-3| + C$$

(p) $\int \ln(y^2+9)\,dy$

SOLUTION: Just like the regular log, we can integrate by parts

$$\begin{array}{c|c|c} + & \ln(y^2+9) & 1 \\ \hline - & \frac{2y}{y^2+9} & y \end{array} \quad \Rightarrow \quad y\ln(y^2+9) - 2\int \frac{y^2}{y^2+9}\,dy$$

For the integral in y, we can use trig substitution: $y = 3\tan(\theta)$ so that $y^2 + 9 = 9(\tan^2(\theta) + 1) = 9\sec^2(\theta)$ and $dy = 3\sec^2(\theta)$:

$$\int \frac{y^2}{y^2+9}\,dy = \int \frac{9\tan^2(\theta)(3\sec^2(\theta))}{9\sec^2(\theta)}\,d\theta = 3\int \tan^2(\theta)\,d\theta$$

Now, use the formulas that will be handed out (about half way down the page) to get that

$$3\int \tan^2(\theta)d\theta = 3\left(\tan(\theta) - \theta\right)$$

Convert back to y so that:

$$-2\int \frac{y^2}{y^2+9}\,dy = -6\cdot\frac{y}{3} + 6\tan^{-1}\left(\frac{y}{3}\right)$$

Put it all together:

$$y\ln(y^2+9) - 2y + 6\tan^{-1}(y/3) + C$$

(q) $\displaystyle\int \frac{\sin^3(x)}{\cos^4(x)}\,dx$

Retain one $\sin(x)$ to go with dx, and set up the substitution $u = \cos(x)$ $du = -\sin(x)\,dx$:

$$-\int (1-u^2)u^{-4}\,du = -\int u^{-4} - u^{-2}\,du = \frac{1}{3}\sec^3(x) - \sec(x) + C$$

(r) $\displaystyle\int \mathrm{e}^{-x}\sin(2x)\,dx$

Integrate by parts twice to get the same integral on both sides,

$+$	$\sin(2x)$	e^{-x}
$-$	$2\cos(2x)$	$-\mathrm{e}^{-x}$
$+$	$-4\sin(2x)$	e^{-x}

Therefore, we have:

$$\int \mathrm{e}^{-x}\sin(2x)\,dx = -\mathrm{e}^{-x}(\sin(2x) + 2\cos(x)) - 4\int \mathrm{e}^{-x}\sin(2x)\,dx$$

and

$$\int \mathrm{e}^{-x}\sin(2x)\,dx = -\frac{1}{5}\mathrm{e}^{-x}(\sin(2x) + 2\cos(x)) + C$$

(s) $\displaystyle\int \frac{w}{\sqrt{w+5}}\,dw$

SOLUTION: After some trial and error, we might take

$$u = \sqrt{w+5}$$

We'll need to solve this for w and dw to make the substitution:

$$w = u^2 - 5 \quad \Rightarrow \quad dw = 2u\,du$$

Therefore,

$$\int \frac{w}{\sqrt{w+5}}\,dw = \int \frac{(u^2-5)2u\,du}{u} = \frac{2}{3}u^3 - 10u + C =$$

$$\frac{2}{3}(w+5)^{3/2} - 10(w+5)^{1/2} + C$$

(t) $\int y^2 \mathrm{e}^{-3y}\,dy$

SOLUTION: Integration by parts using a table

$+$	y^2	e^{-3y}
$-$	$2y$	$(-1/3)\mathrm{e}^{-3y}$
$+$	2	$(1/9)\mathrm{e}^{-3y}$
$-$	0	$(-1/27)\mathrm{e}^{-3y}$

Then just write out the answer. Notice that we can factor out $-\mathrm{e}^{-3y}$ to get:

$$-\mathrm{e}^{-3y}\left(\frac{1}{3}y^2 + \frac{2}{9}y + \frac{2}{27}\right) + C$$

(u) $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\pi/4} \frac{\sec^2(\theta)d\theta}{\sec^4(\theta)} = \int_0^{\pi/4} \cos^2(\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/4} 1 + \cos(2\theta)\,d\theta =$

$$\left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right|_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{4}$$

Notes: Initially, we used $x = \tan(\theta)$, from which we were able to compute the new bounds- If $x = 0$, then $\theta = 0$, and if $x = 1$, then $\theta = \pi/4$.

(v) $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = \int_{36}^{25} \frac{-\frac{1}{2}\,du}{u^{1/2}} = u^{1/2}\Big|_{27}^{36} = 6 - \sqrt{27}$

10. Suppose we have a sphere with radius 3. Set up an integral representing the volume of the cap of the sphere that comes down 1 unit.

    SOLUTION: See attached.

11. Simplify the following expressions (using a triangle):

$$\tan(\sin^{-1}(x/2)) \qquad\qquad \cos(\tan^{-1}(2/x))$$

SOLUTIONS: See attached.

12. If $\sin(\theta) = \frac{x}{\sqrt{x^2+4}}$, find an expression for $\sin(2\theta)$ and $\cos(2\theta)$.
SOLUTIONS: See attached.

13. Suppose that:

x	$f(x)$	$f'(x)$
1	2	5
4	7	3

where f'' is continuous. Find the value of $\displaystyle\int_1^4 x f''(x)\,dx$.

SOLUTION: Using integration by parts:

$$\begin{array}{ccc} + & x & f''(x) \\ - & 1 & f'(x) \\ + & 0 & f(x) \end{array} \quad \to (xf'(x) - f(x)|_1^4 = (f'(1) - f(1)) - (4f'(4) - f(4)) =$$

$$(5-2) - (4 \cdot 3 - 7) = 3 - 5 = -2$$

14. Complete the square:

(a) $2x^2 + 8x = 2(x^2 + 4x + 4) - 8 = 2(x+2)^2 - 8$

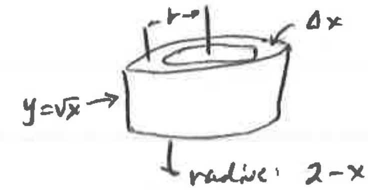
(b) $6 - 3x^2$ **TYPO.** Should have had something like $6 - 3x^2 - x$. In that case, we would write:

$$6 - 3\left(x^2 + \frac{1}{3}x + \frac{1}{36}\right) + \frac{1}{12} = \frac{73}{12} - 3\left(x + \frac{1}{6}\right)^2$$

15. Perform the long division shown. Write the result as we would if we were starting Partial Fraction Decomposition.

$$\frac{x^4 - 4x^3 - 4x^2 - 5x + 4}{x^2 + x + 1} = x^2 - 5x + \frac{4}{x^2 + x + 1}$$

#4) (a) $y=\sqrt{x}$
$y=0$
$x=1$



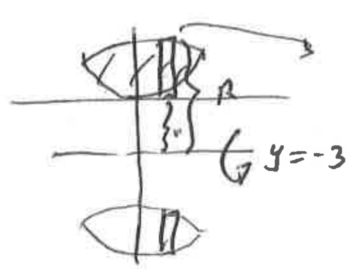
$$V = 2\pi\int_0^1 rh\,dx = 2\pi\int_0^1 (2-x)\sqrt{x}\,dx$$

(b) $y=x^2$
$y=2-x^2$

$$V = 2\pi\int_{-1}^1 rh\,dx = \int_{-1}^1 (1-x)(2-2x^2)\,dx$$

$r = 1-x$
$h = 2-x^2-x^2$

(c) $y=x^2$
$y=2-x^2$
About $y=-3$



Washer

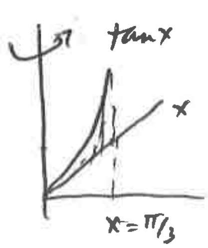
$r = x^2+3$
$R = 2-x^2+3 = 5-x^2$

$$V = \pi\int_{-1}^1 R^2-r^2\,dx = \int_{-1}^1 (5-x^2)^2-(x^2+3)^2\,dx$$

d) $y=\tan(x)$
$y=x$
$-\frac{\pi}{3} \le x \le \frac{\pi}{3}$
↑
Typo

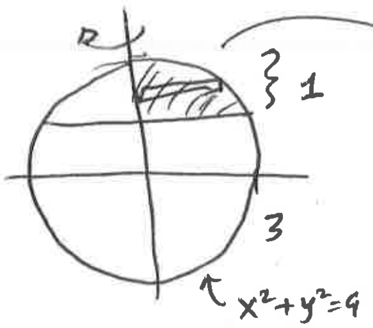
Note: Close to $y=x$, the 2 curves are lines - it is hard to tell them apart, but $\tan(x)$ is slightly larger.

By Symmetry, we need only work in the first quad.



Shells: $2\cdot\int_0^{\pi/3} 2\pi rh\,dx = 2\cdot\int_0^{\pi/3} 2\pi x(\tan x - x)\,dx.$

#10



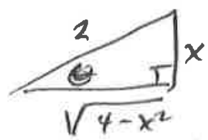
$r = x = \sqrt{9-y^2}$

so $\pi r^2 = \pi(9-y^2)$, and

$$V = \int_2^3 \pi(9-y^2)\,dy$$

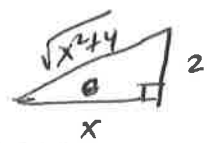
#11

For $\tan(\underbrace{\sin^{-1}(\tfrac{x}{2})}_{\theta}) \longrightarrow$ $\sin\theta = \frac{x}{2}$



$$\boxed{\tan\theta = \frac{x}{\sqrt{4-x^2}}}$$

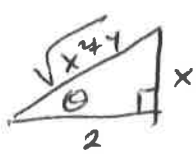
For $\cos(\underbrace{\tan^{-1}(\tfrac{2}{x})}_{\theta}) \longrightarrow \tan\theta = \frac{2}{x}$



so $\boxed{\cos\theta = \frac{x}{\sqrt{x^2+4}}}$

#12

$\sin\theta = \frac{x}{\sqrt{x^2+4}} \rightarrow$



So $\sin(2\theta) = 2\sin\theta\cos\theta = 2\,\frac{x}{\sqrt{x^2+4}} \cdot \frac{2}{\sqrt{x^2+4}} = \frac{4x}{x^2+4}$

$\cos(2\theta) = \cos^2 x - \sin^2 x = \left(\frac{2^2}{x^2+4}\right) - \left(\frac{x^2}{x^2+4}\right) = \frac{4-x^2}{x^2+4}$