Quiz 9 Solutions

Does the series have absolute convergence, conditional convergence, or is it divergent? Be explicit about the test you are using.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)3^n}{2^{2n+1}}$

   SOLUTION: First test for absolute convergence using the Ratio Test:

$$\lim_{n\to\infty}\frac{(n+2)\,3^{n+1}}{2^{2(n+1)+1}}\cdot\frac{2^{2n+1}}{(n+1)\,3^n}=\lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{3}{2^2}=\frac{3}{4}<1$$

   (Note: In the denominator, we have 2^{2n+3}, and the numerator, 2^{2n+1}, which leaves 2^2 in the denominator when we cancel). By the Ratio Test, the series converges absolutely. *See the next page for the much more complicated Root Test.*

2. $\displaystyle\sum_{n=1}^{\infty}\frac{n(n+2)}{(n+3)^2}$

   SOLUTION: This diverges by the test for divergence. You can either divide numerator and denominator by n^2, or use l'Hospital's rule.

$$\lim_{n\to\infty}\frac{n(n+2)}{(n+3)^2}=\lim_{n\to\infty}\frac{2n+2}{2n+6}=1$$

   Our terms do not go to zero, so the series diverges (again, by the test for divergence).

3. $\displaystyle\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{9}}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{2n+1}}$

   SOLUTION: If we look for absolute convergence, we get $\sum\frac{1}{\sqrt{2n+1}}$, which we can compare to the divergent $p-$series: $\sum\frac{1}{\sqrt{n}}$ using the limit comparison test.

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1}{\sqrt{2n+1}}\cdot\frac{\sqrt{n}}{1}=\lim_{n\to\infty}\sqrt{\frac{n}{2n+1}}=\sqrt{\frac{1}{2}}$$

   Therefore, the two series diverge together, and the series does not converge absolutely. On the other hand, the series is alternating, and if we define

$$b_n=\frac{1}{\sqrt{2n+1}},$$

   then b_n is decreasing and the limit is zero. By the Alternating Series Test, the series converges conditionally.

Alternate, much longer solution to #1 using the Root Test. The main problem is that we have to compute the limit of something of the form $f(x)^{g(x)}$, and this is usually done using a "trick": For $A > 0$,

$$A = \mathrm{e}^{\ln(A)}.$$

Or, if we're given $f(x)^{g(x)}$, we can re-express it as a product:

$$f(x)^{g(x)} = \mathrm{e}^{\ln(f(x)^{g(x)})} = \mathrm{e}^{g(x)\ln(f(x))}$$

(See pages 306-307 in our book for more details). This means that

$$\lim_{x\to a} f(x)^{g(x)} = \mathrm{e}^{\lim_{x\to a} g(x)\ln(f(x))}$$

so we typically compute the limit separately:

$$\lim_{x\to a} g(x)\ln(f(x))$$

Using the Root Test in #1 leads us to the following expression:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{(n+1)\,3^n}{2^{2n+1}}\right)^{1/n} = \lim_{n\to\infty} \frac{\sqrt[n]{n+1}\cdot 3}{2^{2+1/n}} = \lim_{n\to\infty} \sqrt[n]{\frac{n+1}{2}} \cdot \frac{3}{2^2}.$$

Therefore, we need to compute the following limit:

$$\lim_{n\to\infty} \left(\frac{n+1}{2}\right)^{1/n} = \lim_{n\to\infty} \mathrm{e}^{(1/n)\ln((n+1)/2)}$$

If we compute the exponent first using l'Hospital's Rule:

$$\lim_{n\to\infty} \frac{\ln((n+1)/2)}{n} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Overall, we now have:

$$\lim_{n\to\infty} \left(\frac{n+1}{2}\right)^{1/n} = \mathrm{e}^0 = 1$$

This is why the Root Test isn't used much...