Summary: Chapter 6

In this chapter, we have used integrals to compute areas and volumes, and how to compute the average value of a function. Additionally, we looked at how to compute work.

In both area and volume computation, we take the "divide and conquer" approach: Approximate the area (volume) by rectangles (disks, washers or shells), then add them together. Finally, the approximation becomes exact when we integrate.

1. Area. Here, rather than measuring the height of a rectangle from an axis, the height ranges between two functions. Therefore, the area is given by:

$$\int_a^b |f(x) - g(x)| \text{ dx} \qquad \int_a^b |f(y) - g(y)| \text{ dy}$$

   Choose integration with respect to x or y depending on how easy it will be to get an expression for the heights. We interpret this integral as the "Upper function − Lower function", where f and g may switch roles. When integrating in y, "Upper" means "rightmost".

2. Volume. We used three basic shapes, and their use will depend on what cross sections of the volume look like. You should always sketch what one of your "slabs" look like, and analyze what the dimensions will be from that. If you're not sure, think about what happens for sample numerical values (i.e., "what will the radius be if $x = 1$? What will the radius be if $y = 1$?, etc.). The three basic volumes are given below. The overall volume is then found by integrating over these samples.

   (a) Disk: $\pi r^2 h$
   (b) Washer: $\pi(r_2^2 - r_1^2)h$ (Remember this as the full disk minus the small disk).
   (c) Shell: If the radius is r, the width is w (w=dx or w=dy), and the height is h, then: $2\pi r h w$ (Circumference · height · width)

3. Work. Keep in mind the following table:

   mass (kg)
   Force (N)
   accel(m/s^2)
   Work (J)
   Dist (m)

   When working with water, we add:

   Vol (m^3)
   mass (kg)
   Density (kg/m^3)
   Force (N)
   accel(m/s^2)
   Work (J)
   Dist (m)

   Note that in U.S. Customary units, this chart becomes simpler:

   Vol (ft^3)
   Force (lbs)
   Density (lbs/ft^3)
   Work (ft-lbs)
   Dist (ft)

   Additionally, we have Hooke's Law (for springs), which states that the force it takes to stretch a spring x units beyond its natural length is proportional to x. That is, $f(x) = kx$.

4. Average Value. This is an application of the Mean Value Theorem for derivatives:

   Let f be a continuous function on $[a, b]$. Then there exists a value c in $[a, b]$ such that:

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x) \text{ dx}$$

   Heuristically (for f positive), the height $f(c)$ is the height at which we would cut off the mountaintops to fill the valleys. Or (again for f positive), the area under the curve is the same as the area of a rectangle $f(c)$ units high (and $b - a$ units wide).