Summary: Techniques of Integration

We've had 5 basic integrals that we have developed techniques to solve:

1. Integration by parts: Three basic problem types: (1) $x^n f(x)$: Use a table, if possible. (2) Exponential times a sine or cosine: Integrate by parts twice to get the same integral type on both sides of the equation. (3) Some functions, like $\sin^{-1}(x)$ and $\ln(x)$, have special derivatives. When integrating these types of functions, use integration by parts once, with $dv = dx$.

2. Trig integrals: Two techniques- (1) Try to keep something with dx and make a u, du substitution. (2) Use half-angle identities to write powers of sines and cosines as $\sin(mx)$ and $\cos(mx)$, which can be integrated directly.

   (a) Odd power of sine or cosine: Try u, du.

   (b) Both are even powers: Use half-angle identities.

   (c) Integrals with other trig functions: First, try to keep out $\sec(x)\tan(x)$ or $\sec^2(x)$ with the dx to get a substitution. If that doesn't work, try writing in terms of sines and cosines to get something that does work.

3. Trig substitutions: The idea here is to substitute trig functions in for x to get an integral for which we can use the techniques developed in 7.2. Templates:

$\sqrt{a^2-x^2}$	$x = a\sin(\theta)$	$\sqrt{a^2-c^2x^2}$	$cx = a\sin(\theta)$	$\sqrt{a^2-(x-b)^2}$	$x-b = a\sin(\theta)$
$\sqrt{x^2+a^2}$	$x = a\tan(\theta)$	$\sqrt{c^2x^2+a^2}$	$cx = a\tan(\theta)$	$\sqrt{(x-b)^2+a^2}$	$x-b = a\tan(\theta)$
$\sqrt{x^2-a^2}$	$x = a\sec(\theta)$	$\sqrt{c^2x^2-a^2}$	$cx = a\sec(\theta)$	$\sqrt{(x-b)^2-a^2}$	$x-b = a\sec(\theta)$

   Note: To get an expression into the form $(x-b)^2 - a^2$, $(x-b)^2 + a^2$ or $a^2 - (x-b)^2$, we usually have to complete the square.

   What would we substitute for $\sqrt{4(x+1)^2 - 3}$? [1]

4. Partial Fractions: In this case, we have a polynomial, $P(x)$ divided by a polynomial $Q(x)$, and the degree of P is less than the degree of Q (If this is not the case, do long division). Our goal is to write the fraction, with the factors of Q, as a sum of simpler fractions, each one having a type of factor from Q. We can summarize this technique with the following table:

$Q(x)$ has a factor like:	The sum has a term like:
$(ax+b)$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b)} + \ldots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Bx+C}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{B_1x+C_1}{ax^2+bx+c} + \ldots + \frac{B_kx+C_k}{(ax^2+bx+c)^k}$

   Remember to solve for the constants by multiplying both sides of the equation by the denominator, then set x to convenient values.

[1] $4(x+1) = \sqrt{3}\tan(\theta)$

5. Improper Integrals. The key idea here is that an improper integral is a limit. There were two types of integrals- One type had ∞ appearing as an integral bound, the other type occurred if the integrand had an infinite discontinuity:

$$\int_a^\infty f(x)\ dx = \lim_{t\to\infty}\int_a^t f(x)\ dx$$
$$\int_{-\infty}^a f(x)\ dx = \lim_{t\to-\infty}\int_t^a f(x)\ dx$$

If f has a vertical asymptote at $x = a$:

$$\int_a^b f(x)\ dx = \lim_{t\to a^+}\int_t^b f(x)\ dx$$

If f has a vertical asymptote at $x = b$:

$$\int_a^b f(x)\ dx = \lim_{t\to b^-}\int_a^t f(x)\ dx$$

Some techniques from the past to recall:

- L'Hospital's rule. (For limits like te^t, $t\ln(t)$, etc.)
- Logarithm rules: $\ln(a) + \ln(b) = \ln(ab)$, $\ln(a) - \ln(b) = \ln(\frac{a}{b})$.
- Computing horizontal asymptotes.

6. One last technique that might be useful: Given $\sqrt[n]{g(x)}$, you might try $u = \sqrt[n]{g(x)}$, so that $u^n = g(x)$, and $nu^{n-1}\ du = g'(x)\ dx$.

7. Formula Table to be given to you on the exam:

- $\int \cot(x)\ dx = \ln|\sin(x)| + C$
- $\int \tan(x)\ dx = \ln|\sec(x)| + C$
- $\int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C$
- $\int \csc(x)\ dx = \ln|\csc(x) - \cot(x)| + C$
- $$\begin{aligned}\sin(A)\cos(B) &= \tfrac{1}{2}[\sin(A-B) + \sin(A+B)]\\ \sin(A)\sin(B) &= \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]\\ \cos(A)\cos(B) &= \tfrac{1}{2}[\cos(A-B) + \cos(A+B)]\end{aligned}$$
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) = 1 = 1 - 2\sin^2(x)$

8. Note that you should remember important trig identities like:

$$\begin{aligned}\sin^2(x) + \cos^2(x) &= 1\\ \tan^2(x) + 1 &= \sec^2(x)\\ \sin(2x) &= 2\sin(x)\cos(x)\\ \cos^2(x) &= \tfrac{1}{2}[1 + \cos(2x)]\\ \sin^2(x) &= \tfrac{1}{2}[1 - \cos(2x)]\end{aligned}$$