Final Exam Review Solutions

1. (6.2, 32) Set up an integral for the volume of the solid obtained by rotating the region defined by $y = \sqrt{x-1}$, $y = 0$ and $x = 5$ about the $y-$axis.

$$V = \int_0^2 \pi(5^2 - (y^2+1)^2)\ dy$$

2. (5.1, 16) Write the area under $y = \sqrt[3]{x}$, $0 \leq x \leq 8$ as the limit of a Riemann sum (use right endpoints).

$$\Delta x = \frac{8}{n},\ \text{Height}_i = \sqrt[3]{\frac{8i}{n}} \Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[3]{\frac{8i}{n}} \cdot \frac{8}{n}$$

3. (6.2, 14) Find the volume of the solid obtained by rotating the region bounded by: $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 3$ about $y = -1$.

$$V = \int_1^3 \pi \left[\left(\frac{1}{x}+1\right)^2 - 1^2\right]\ dx$$

4. (6.2, 40) The integral $\pi\int_2^5 y\ dy$ represented the volume of a solid. Describe the solid.

   Rotate the region to the left of $\sqrt{y}$ and to the right of the $y - axis$ about the $y-$axis $(2 \leq y \leq 5)$.

5. (5.1, 18) Determine a region whose area is equal to the following limit (do not evaluate the limit): $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$

   The region is the area under the curve $y = \sqrt{x}$ for $1 \leq x \leq 4$. You could also say that its under the curve $y = \sqrt{1+x}$ for $0 \leq x \leq 3$.

6. (5.2, 15) Write the following limit as a definite integral on the given interval: $\lim_{n\to\infty}\sum_{i=1}^{n} x_i \sin(x_i)\Delta x$, $[0, \pi]$.

$$\int_0^{\pi} x\sin(x)\ dx$$

7. (6.1, 17) Find the area between the curves $y^2 = x$ and $x - 2y = 3$.

   Taking horizontal rectangles (integrate with respect to y), we see that the rightmost function is $x-2y = 3$. The points of intersection are $(1, -1)$ and $(9, 3)$. The area is:

$$\int_{-1}^{3} (2y+3) - y^2\ dy = \frac{32}{3}$$

8. (5.2, 44) Write the following difference as a single integral: $\int_2^{10} f(x)\ dx - \int_2^7 f(x)\ dx$

$$\int_7^{10} f(x)\ dx$$

9. (5.2, 46) If $\int_0^1 f(x)dx = 2, \int_0^4 f(x)dx = -6$, and $\int_3^4 f(x)\ dx = 1$, find $\int_1^3 f(x)\ dx$. $-6-2-1 = -9$

10. (Similar to 5.2, 39) If $\int_0^1 f(x)\ dx = \frac{1}{3}$, what is $\int_0^1 5 - 6f(x)\ dx$? $5(1) - 6\frac{1}{3} = 5 - 2 = 3$.

11. (5.3, 9) Compute $\frac{dF}{dx}$, if $F(x) = \int_x^2 \cos(t^2)\ dt$ Note first that $\int_x^2 \cos(t^2)dt = -\int_2^x \cos(t^2)\ dt$, which is now in standard form.

$$-\cos(x^2)$$

12. (5.3, 13) Compute $\frac{dg}{dy}$, if $g(y) = \int_3^{\sqrt{y}} \frac{\cos(t)}{t}\ dt$.

$$\frac{cos(\sqrt{y})}{\sqrt{y}} \cdot \frac{1}{2\sqrt{y}}$$

13. (5.3, 24 and 7.8) Evaluate: $\int_{-1}^{1} 3t^{-4}\ dt$ We note that this integral is improper:

$$\lim_{T\to 0^-}\int_{-1}^{T} \frac{3}{t^4}\ dt + \lim_{T\to 0^+}\int_{T}^{1} \frac{3}{t^4}\ dt$$

   and neither of these limits exist.

14. (5.3, 50) Find $\frac{dy}{dx}$, if $y = \int_{\cos(x)}^{5x} cos(t^2)\ dt$

   First, in standard form, (0 was convenient, use any constant):

$$y = -\int_0^{\cos(x)} \cos(t^2)\ dt + \int_0^{5x} \cos(t^2)\ dt$$

   so the derivative is:

$$\frac{dy}{dx} = \cos(\cos^2(x)) \cdot \sin(x) + \cos(25x^2) \cdot 5$$

15. (5.3, 58) Evaluate the limit by recognizing the sum as a Riemann sum:

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \ldots + \sqrt{\frac{n}{n}}\right)$$

$$\int_1^4 \frac{1}{\sqrt{x}}\ dx = 2$$

16. (5.2, 22) Write the following integral as the limit of a Riemann sum (use right endpoints): $\int_0^5 (1+2x^3)\ dx$

$$\lim_{n\to\infty}\sum i = 1^n \left(1 + \left(\frac{5i}{n}\right)^3\right)\frac{5}{n}$$

17. (5.2, 37) Given that $\int_4^9 \sqrt{x}\quad dx = \frac{38}{3}$, what is $\int_9^4 \sqrt{t}\ dt$? $-38/3$

18. (6.5, 10) Let $f(x) = \mathrm{e}^x$ on the interval $[0, 2]$. (a) Find the average value of f. (b) Find c such that $f_{\text{avg}} = f(c)$.

$$f_{\text{avg}} = \frac{1}{2}\int_0^2 \mathrm{e}^x\ dx = \frac{1}{2}(\mathrm{e}^2 - 1)$$

$$\mathrm{e}^c = \frac{\mathrm{e}^2 - 1}{2} \Rightarrow c = \ln\left(\frac{\mathrm{e}^2-1}{2}\right)$$

19. (5.4, 53) The velocity function is $v(t) = 3t - 5$. (a) Find the displacement, (b) Find the distance traveled. Displacement for $t = a$ to $t = b$ is $\int_a^b 3t-5\ dt$ Distance is $\int_a^b |3t-5|\ dt$

20. Exercise 7, pg. 427 (There are some graphs to consider). See the back of the book.

21. (Ch 5 Review, 68) Suppose $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, and $h''(2) = 13$, and h'' is continuous. Evaluate $\int_1^2 h''(u)\ du$. $h'(2) - h'(1) = 5 - 2 = 3$.

22. (6.1, 24) Find the area between the curves $y = |x|$ and $y = x^2 - 2$. (Also was a HW problem).

$$\int_{-2}^{2} |x| - (x^2 - 2)\ dx = 2\int_0^2 x - (x^2 - 2)\ dx = \frac{20}{3}$$

23. (5.5, 78) If f is continuous and $\int_0^9 f(x)\ dx = 4$, find $\int_0^3 xf(x^2)\ dx$. Let $u = x^2$, so $du = 2x\ dx$ Then:

$$\int_0^3 xf(x^2)\ dx = \frac{1}{2}\int_0^9 f(u)\ du = 2.$$

24. Problem 41, p. 439 (Pictures) See the back of the book.

25. (6.1, 46(a)) Find a so that half the area under the curve $y = \frac{1}{x^2}$ lies in the interval $[1, a]$ and half of the area lies in the interval $[a, 4]$.

$$\int_1^4 \frac{1}{x^2}\ dx = 2\int_1^a \frac{1}{x^2}\ dx$$

$$1 - \frac{1}{4} = 2 - \frac{2}{a} \Rightarrow a = \frac{8}{5}$$

26. (6.3, 22) Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by $y = x$, $y = 4x - x^2$, about $x = 7$. (Shells)

$$\int_0^3 2\pi(7 - x)((4x - x^2) - x)\ dx$$

27. (CH 8 review, 11) Find the centroid of the region bounded by the curves $y = 4-x^2$ and $y = x+2$. Area is $9/2$, M_y is $-9/4$, M_x is $54/5$. Overall, $\bar{x} = -\frac{1}{2}$, $\bar{y} = \frac{12}{5}$.

28. (6.4 17-20) Review the Work problems, 17-20, p. 459. See the web site for the writeups.

29. (6.5, 13) If f is continuous, and $\int_1^3 f(x)dx = 8$, show that f take on the value 4 at least once in the interval $[1, 3]$. The average value of f is $8/2 = 4$. By the Mean Value Theorem for Integrals (avg value formula), this means that there is a c in $[1, 3]$ such that $f(c) = 4$.

30. (Ch 6 Review, 16) Let R be the region in the first quadrant bounded by $y = x^3$ and $y = 2x - x^2$. Calculate: (a) The area of R (5/12) (b) Volume obtained by rotating R about the $x-$axis ($41\pi/105$) (c) Volume obtained by rotating R about the $y-$axis ($13\pi/30$), (d) The centroid of R.

    We did this problem in class. Numerical values are provided.

31. (Ch 8 Review, 13 and 14) Find the centroid: (13:) $(2, 2/3)$, (14:) $(9/10, 3/2)$

32. (Ch 8 Review, 3) Set up an integral to find the arc length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \le x \le 2$. You should get that

$$1 + (y')^2 = \frac{1}{4}(x^4 + 2 + x^{-4}) = \frac{1}{4}(x^2 + x^{-2})^2$$

so that the arc length is:

$$\int_1^2 \frac{1}{2}(x^2 + x^{-2}\ dx$$

33. What does the Theorem of Pappus say? If we have a region R with area A that lies completely to one side of line l, then the volume of the solid of revolution about l is A times the distance that the centroid travels.

34. (Similar to Ch 5 Review, 8(b)) Evaluate: $\frac{d}{dx}\int_0^1 x^2 \sin(x)\quad dx$ The derivative is zero, since the definite integral is a number. (The derivative of a constant is zero).

35. (Ch 8.3, Example 6, see text) Find the centroid of the region bounded by the curves $y = x^2$ and $y = x$.

36. Write the appropriate partial fraction expansion for the following expression (do not solve for the constants):

$$\frac{1+x}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

37. If an integral has $\sqrt{2x - x^2}$, what is the appropriate trigonometric substitution? Complete the square first:

$$2x - x^2 = -(x^2 - 2x + 1) + 1 = 1 - (x-1)^2$$

so substitute $x - 1 = \sin(\theta)$

38. If an integral has $\sqrt{1 - 2(x+3)^2}$, what is the appropriate trigonometric substitution? Substitute $\sqrt{2}(x+3) = \sin(\theta)$.

39. What is the derivative of $\sin^{-1}(x)$? $\tan^{-1}(x)$? The derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$. The derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$

40. Let f be differentiable. Integrate by parts: $\int f(x)\ dx$.

$$xf(x) - \int xf'(x)\ dx$$

41. Suppose you are integrating $P(x)/Q(x)$, where P and Q are polynomials. Explain the process by which we integrate this expression. (Consider the degrees of P and Q). First, if the degree of $P \ge$ the degree of Q, perform long division. We now can assume that the degree of P is less than the degree of Q. Factor Q completely, and use partial fractions.

42. What was the Mean Value Theorem for Integrals? The same as the average value formula. If f is continuous on the interval $[a, b]$, then there is a c in the interval such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

43. True or False: Every elementary function has an antiderivative. If it is continuous, this is true (Fund. Theorem of Calc). However, the we may not be able to write the antiderivative as a "simple expression" from a table.

44. If we have $\int_a^b f(x)dx$, and $f(x)$ has a vertical asymptote at c, where c is in the interval $[a, b]$, explain how we proceed. Proceed by taking two integrals and performing a limit as each approaches c (See a similar previous problem, $\int_{-1}^{1} \frac{3}{t^4}\,dt$.)

45. Evaluate: $\int_0^\infty t\mathrm{e}^{-st}\,dt$, where s is a positive constant. Integration by parts (use a table) gives:

$$\lim_{T\to\infty} \frac{-t}{s}\mathrm{e}^{-st}\Big|_0^T + \lim_{T\to\infty} \frac{-1}{s^2}\mathrm{e}^{-st}\Big|_0^T$$

Use L'Hospital's rule to compute the limits to get that the first term is 0, and the overall result is $1/s^2$.

46. What does L'Hospital's rule say? See page:

47. Evaluate:

(a) (Ch 7 Review, 4) $\displaystyle\int \frac{\sec^2(x)}{1-\tan(x)}\,dx$ Let $u = \tan(\theta)$. $-\ln|1-\tan(\theta)| + C$

(b) (Ch 5 Review, 20) $\displaystyle\int_{-1}^{1} \frac{x^5+x^3+x}{x^4+x^2+1}\,dx$ Note that simplifying the integrand gives: $\int_{-1}^{1} x\,dx = 0$.

(c) (Ch 7 Review, 7) $\displaystyle\int \tan^7 x \sec^3 x\ \ dx$ Keep a $sec(x)\tan(x)$ out with dx, and take $u = \sec(x)$. You get $\int (u^2-1)^3u^2du$, which we have to multiply out. Final answer: $\frac{1}{9}\sec^9(x) - \frac{3}{7}\sec^7(x) + \frac{3}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C$

(d) (5.5, 31) $\displaystyle\int \frac{dx}{x\ln(x)}$ Let $u = \ln(x)$, so $\ln|\ln|x|| + C$

(e) (Ch 7 Review, 6) $\displaystyle\int \frac{1}{y^2-4y-12}\,dy$ Use partial fractions to get $\frac{1}{8}\ln|y-6| - \frac{1}{8}\ln|y+2| + C$

(f) (5.4, 9) $\displaystyle\int (1-t)(2+t^2)\,dt$ Multiply it out, $2t - t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + C$

(g) (5.4, 25) $\displaystyle\int u(\sqrt{u} + \sqrt[3]{u})\,du$ Simplify before integrating. $\frac{2}{5}u^{5/2} + \frac{3}{7}u^{7/3} + C$

(h) (Ch 7 Review, 33) $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,dx$ 1/36.

(i) (5.4, 28) $\displaystyle\int_{-1}^{2} |x-x^2|\,dx$ Split the integral like we do in area problems: $\int_{-1}^{0} x^2 - x\,dx + \int_0^1 x - x^2\,dx + \int_1^2 x^2 - x\,dx$ to get 11/6

(j) (Ch 7 Review, 36) $\displaystyle\int_1^4 \frac{\mathrm{e}^{1/x}}{x^2}\,dx$ Let $u = 1/x$ and substitute. $e - e^{1/4}$

(k) (Ch 7 Review, 29) $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$ Let $x = 2\sin(\theta)$ and substitute.

$$\frac{x}{\sqrt{4-x^2}} - \sin^{-1}(x/2) + C$$

(l) (Ch 7 Review, 26) $\displaystyle\int \frac{1}{1+\mathrm{e}^x}\,dx$ Let $u = \mathrm{e}^x$, so $\ln(u) = x$ and $(1/u)du = dx$ and substitute. You will then need to do partial fractions, and get:

$$x - \ln(1+\mathrm{e}^x) + C$$

(m) (5.5, 22) $\displaystyle\int \frac{\tan^{-1}(x)}{1+x^2}\,dx$ Let $u = \tan^{-1}(x)$. $\frac{1}{2}(\tan^{-1}(x))^2 + C$

(n) (5.5, 70) $\displaystyle\int_{-a}^{a} x\sqrt{x^2-a^2}\,dx$ You should get 0.

(o) (5.5, 56 (+section 7.8)) $\displaystyle\int_0^2 \frac{dx}{(2x-3)^2}\,dx$ Does not exist (take the limit as $T \to \frac{3}{2}$).

(p) (Ch 7 Review, 13) $\displaystyle\int \sin^2 x \cos^5 x\,dx$ Pull out a $\cos(x)$ to keep with dx.

$$\frac{1}{3}\sin^3(x) - \frac{2}{5}\sin^5(x) + \frac{1}{7}\sin^7(x) + C$$

(q) (5.5, 12) $\displaystyle\int \frac{x}{x^2+1}\,dx$ $\frac{1}{2}\ln(x^2+1) + C$

(r) (Ch 7 Review, 19) $\displaystyle\int \frac{1}{\sqrt{x^2-4x}}\,dx$ First complete the square then substitute $x - 2 = \sec(\theta)$. You should end up with $\int \sec(\theta)\,d\theta$, which will be given on the exam... Be sure you can go back to x.

(s) (Ch 7.5, 36) $\displaystyle\int \sin(4x)\cos(3x)\,dx$

(t) (Ch 7 Review, 5) $\displaystyle\int x^4\ln(x)\,dx$ Integrate by parts with $u = \ln(x)$, $dv = x^4\,dx$.

$$\frac{1}{5}x^5\ln(x) - \frac{1}{25}x^5 + C$$