Homework Solutions: Chapter 7

Section 7.1

1. Integrate by parts:

$$u = \ln(x) \qquad dv = x\,dx$$
$$du = \tfrac{1}{x}\,dx \qquad v = \tfrac{1}{2}x^2$$

so that

$$\int x\ln(x)\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C$$

3. Use a table: $\int xe^{2x}\,dx$ is:

$+$	x	e^{2x}
$-$	1	$\frac{1}{2}e^{2x}$
$+$	0	$\frac{1}{4}e^{2x}$

$$\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

5. Use a table: $\int x\sin(4x)\,dx$ is:

$+$	x	$\sin(4x)$
$-$	1	$-\frac{1}{4}\cos(4x)$
$+$	0	$-\frac{1}{16}\sin(4x)$

$$-\frac{1}{4}x\cos(4x) + \frac{1}{16}\sin(4x) + C$$

6. Let

$$u = \sin^{-1}(x) \qquad dv = dx$$
$$du = \tfrac{1}{\sqrt{1-x^2}} \qquad v = x$$

$$x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Now let $u = x^2, du = 2x\,dx$, and the final result is:

$$x\sin^{-1}x + (1-x^2)^{-1/2} + C$$

8. Use a table: $\int x^2\sin(ax)\,dx$ is:

$+$	x^2	$\sin(ax)$
$-$	$2x$	$-\frac{1}{a}\cos(ax)$
$+$	2	$-\frac{1}{a^2}\sin(ax)$
$-$	0	$\frac{1}{a^3}\cos(ax)$

$$-\frac{x^2}{a}\cos(ax) + \frac{2x}{a^2}\sin(ax) + \frac{2}{a^3}\cos(ax) + C$$

10. Use a table: $\int t^2e^t\,dt$ is:

$+$	t^3	e^t
$-$	$3t^2$	e^t
$+$	$6t$	e^t
$-$	6	e^t
$+$	0	e^t

$$e^t\left(t^3 - 3t^2 + 6t - 6\right)$$

24.

$$u = \tan^{-1}(x) \qquad dv = x\,dx$$
$$du = \tfrac{1}{1+x^2} \qquad v = \tfrac{1}{2}x^2$$

$$\frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

By long division, the integral above becomes:

$$\int 1 - \frac{1}{1+x^2}\,dx$$

so the solution is:

$$\frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}x + \frac{1}{2}\tan^{-1}(x) + C$$

28. $\int_0^t e^s\sin(t-s)\,ds$ There are several ways you could've gone about this problem. Note that in any event, we'll use a technique like Example 4, p. 472.

$+$	$\sin(t-s)$	e^s
$-$	$-\cos(t-s)$	e^s
$+$	$-\sin(t-s)$	e^s

$$2\int_0^t e^s\sin(t-s)\,ds = e^s(\sin(t-s) + \cos(t-s))$$

so that the final answer is:

$$\frac{1}{2}e^t - \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t)$$

52. Using shells, the volume is:

$$\int_0^1 2\pi x(e^x - e^{-x})\,dx$$

Note that we have to integrate xe^x and xe^{-x} by parts to get a final answer of $\frac{4\pi}{e}$

Section 7.2

3. Pull out a $\cos(x)$ to put with dx, and make a u, du substitution. You should get:

$$\int_1^{\frac{1}{\sqrt{2}}} u^5(1-u^2)\,du$$

(NOTICE the bounds!) The answer is then $\frac{-11}{384}$.

4. Pull out a $\cos(x)$ to put with dx, and make a u, du substitution. You should get:

$$\int_0^1 (1-u^2)^2\ du$$

for a final answer of $\frac{8}{15}$

15. Pull out a $\sin(x)$ to keep with dx, and do a u, du substitution to get:

$$\int (1-u^2)u^{1/2}(-1)\ du$$

Switching back to x, we get:

$$\frac{2}{7}(\cos(x))^{7/2} - \frac{2}{3}(\cos(x))^{3/2} + C$$

18. Rewriting in terms of sines and cosines, you should get:

$$\int \frac{\cos^5(\theta)}{\sin(\theta)}\ d\theta$$

Now pull out a $\cos(\theta)$ to put with $d\theta$, do a u, du subsitution to get:

$$\int \frac{(1-u^2)^2}{u}\ du$$

Converting back to θ, we get:

$$\ln|\sin(\theta)| - \sin^2(\theta) + \frac{1}{4}\sin^4(\theta) + C$$

41. Use Table 2 to get

$$\frac{1}{2}\int \cos(3x) - \cos(7x)\ dx$$

so that the answer is:

$$\frac{1}{6}\sin(3x) - \frac{1}{14}\sin(7x) + C$$

42. Again use Table 2 to get

$$\frac{1}{2}\int \sin(4x) + \sin(2x)\ dx$$

so that the answer is:

$$-\frac{1}{8}\cos(4x) - \frac{1}{4}\cos(2x) + C$$

52. First, we'll list the answers:

   (a) $-\frac{1}{2}\cos^2(x) + C_1$
   (b) $\frac{1}{2}\sin^2(x) + C_2$
   (c) $-\frac{1}{4}\cos(2x) + C_3$
   (d) $\frac{1}{2}\sin^2(x) + C_4$

   Using the identities: $\cos^2(x) = 1 - \sin^2(x)$ and $\cos(2x) = 1 - 2\sin^2(x)$, we can see that all of the functions above are all shifted (up and down) versions of $\frac{1}{2}\sin^2(x)$. (Which is what we mean when we say "Any two antiderivatives of the same function differ by a constant")

54. Break up the integral at $\frac{\pi}{6}$ and evaluate. You get:

$$1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

60. The volume is (by washers):

$$\int_0^{\pi/2} \pi\left(1^2 - (1-\cos(x))^2\right)\ dx$$

Integrating (be sure to change the bounds), we get:

$$2\pi - \frac{\pi^2}{4}$$

Section 7.3

3. Let $x = 3\tan(\theta)$. Be sure to put in $dx = 3\sec^2(\theta)\ d\theta$. Then the integral should simplify to:

$$3^3\int \tan^3(\theta)\sec(\theta)\ d\theta$$

We have some options now. We can either take out a $sec(\theta)\tan(\theta)$ to keep with $d\theta$, or write everything in sines and cosines first, then try a u, du subsitution. Either way, the answer is:

$$\frac{1}{3}(x^2+9)^{3/2} - 9(x^2+9)^{1/2} + C$$

4. Let $x = \sin(\theta)$. Then $dx = \cos(\theta)\ d\theta$, and the integral simplifies to:

$$4^3\int_0^{\pi/3} \sin^3(\theta)\ d\theta$$

(NOTE the bounds!) Converting to u, du we get:

$$-4^3\int_1^{1/2}(1-u^2)\ du$$

so the answer is: $\frac{40}{3}$

NOTE: You could've also avoided trig substitution altogether by initially letting $u = 16 - x^2$, so $x^2 = 16 - u$, and $du = -2x\ dx$

9. (NOTE: I will give you $\int csc(x)\ dx$ on the exam): Let $x = \sqrt{3}\tan(\theta)$. Then $dx = \sqrt{3}\sec^2(\theta)\ d\theta$ Then the integral simplifies to:

$$\frac{1}{\sqrt{3}}\int \csc(\theta)\ d\theta = \frac{1}{\sqrt{3}}\ln|\csc(\theta) - \cot(\theta)| + C$$

and converting back to x (use a triangle),

$$\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2+3}-\sqrt{3}}{x}\right|+C$$

10. Let $x = a\cdot\sec(\theta)$ (remember to substitute for dx), and we get (after simplification):

$$\frac{1}{a^2}\int \sin^2(\theta)\cos(\theta)\,d\theta$$

integrating and going back to x, we get:

$$\frac{(x^2-a^2)^{3/2}}{3a^2x^3}+C$$

23. Complete the square to get things in the right form:

$$2x-x^2 = 1-(x-1)^2$$

So that $x-1 = \sin(\theta)$ and simplify to $\int \cos^2(\theta)d\theta$. Integrating and substituting $sin(2\theta) = 2\sin(\theta)\cos(\theta)$, then using a triangle, we get:

$$\frac{1}{2}\sin^{-1}(x-1)+\frac{1}{2}(x-1)\sqrt{2x-x^2}+C$$

26. Again, we first complete the square,

$$4x-x^2 = 4-(x-2)^2$$

so that the integral becomes

$$\int \frac{x^2}{\sqrt{4-(x-2)^2}}\,dx$$

Then $x-2 = 2\sin(\theta)$, $x^2 = (2\sin(\theta)+2)^2$, and $dx = 2\cos(\theta)\,d\theta$, so simplifying after substitution will give:

$$4\int(\sin^2(\theta)+2\sin(\theta)+1)\,d\theta$$

To go back to x, recall that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, and use a triangle. The answer is:

$$6\sin^{-1}\left(\frac{x-2}{2}\right)-4\sqrt{4x-x^2}-\frac{x-2}{2}\sqrt{4x-x^2}+C$$

34. (If you need a brush up, Hyperbolas are graphed in Appendix C, page A20) From the picture of a hyperbola, we get that the area is given by:

$$2\int_2^3 \frac{3}{2}\sqrt{x^2-4}\,dx$$

Subsitution of $x = 2\sec(\theta)$ (remember to substitute for dx!) gives (when simplified, and leaving off the bounds):

$$12\int \tan^2(\theta)\sec(\theta)d\theta$$

There are a number of things we can do here, here is one approach:

$$12\int(\sec^2(\theta)-1)\sec(\theta)d\theta$$

$$12\int \sec^3(\theta)-\sec(\theta)\,d\theta$$

The integral of $\sec(x)$ would be provided for you, and the integral of $\sec^3(x)$ is done in Example 8, p. 481. Simplifying, we get:

$$6[\sec(\theta)\tan(\theta)-\ln|\sec(\theta)+\tan(\theta)|]$$

Converting back to x and applying the bounds:

$$6\left[\frac{x}{\sqrt{x^2-4}}-\ln\left|\frac{x}{2}+\frac{2}{\sqrt{x^2-4}}\right|\right]\Bigg|_2^3$$

which is approximately 4.288

Section 7.4

3.

$$\frac{A}{2x+3}+\frac{B}{x-1}$$

4.

$$\frac{A}{3z+5}+\frac{B}{(3z+5)^2}+\frac{C}{(3z+5)^3}+\frac{D}{z+2}$$

10.

$$\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}+\frac{Dx+E}{x^2+1}+\frac{Fx+G}{2x^2+5x+7}+\frac{Hx+I}{2x^2+5x+7)^2}$$

17. First, note that:

$$\frac{x^2+1}{x^2-1} = 1+\frac{x+1}{x(x-1)}$$

After partial fractions, we integrate:

$$\int 1-\frac{1}{x}+\frac{2}{x+1}\,dx$$

which gives:

$$x-\ln|x|+2\ln|x-1|+C = x+\frac{(x-1)^2}{|x|}+C$$

18. Two cases: If $a = b$, and $a \neq b$.

If $a = b$, then we have $\int \frac{dx}{(x+a)^2}$ in which case, we integrate by letting $u = x+a$ and do a u, du substitution which gives:

$$-\frac{1}{x+a}+C$$

If $a \neq b$, then partial fractions gives:

$$\frac{1}{b-a} \int \frac{1}{x+a} - \frac{1}{x+b}\, dx$$

and the answer is:

$$\frac{1}{b-a}(\ln|x+a| - \ln|x+b|) + C = \frac{1}{b-a} \ln\left|\frac{x+a}{x+b}\right| + C$$

21. Partial Fractions gives:

$$\int_1^2 \frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3}\, dy$$

so the final answer is (after simplification) $\frac{9}{5} \ln \frac{8}{3}$

22. Partial fractions gives:

$$\int_2^3 \frac{-1/2}{x} + \frac{1/6}{x+2} + \frac{1/3}{x-1}\, dx$$

the final answer is

$$\ln\left(\sqrt{\frac{2}{3}}\right) + \frac{1}{6}\ln(5) \approx 0.6551$$

29. Perform long division first- Then integrate:

$$\int_0^1 x - \frac{x}{x^2+1}\, dx$$

so the final answer is $\frac{1}{2} - \frac{1}{2}\ln(2)$

39. First, the partial fractions gives:

$$\frac{-1}{x^2} + \frac{1/2}{x-1} - \frac{1/2}{x+1}\, dx$$

The final answer (after simplification) is:

$$\frac{1}{x} + \frac{1}{2}\left|\frac{x-1}{x+1}\right| + C$$

44. Let $u = \sqrt{x+1}$, so $x = u^2 - 1$ and $dx = 2u\, du$. Therefore, we get:

$$\int \frac{2u}{(u^2-1)u}\, du = 2\int \frac{du}{u^2-1}$$

and the final answer is:

$$\ln|\sqrt{x+1} - 1| - \ln|\sqrt{x+1} + 1| + C$$

46. Let $u = \sqrt[3]{x}$, so $u^3 = x$ and $3u^2\, du = dx$. After substitution, we get:

$$\int_0^1 \frac{3u^2}{1+u}\, du$$

After long division and integration, the final answer is

$$3\left(\ln(2) - \frac{1}{2}\right)$$

62. Complete the square, $x^2 - 6x + 8 = (x-3)^2 - 1$ which is positive for $5 \leq x \leq 10$, so (subsituting $u = x - 3$),

$$\int_2^7 \frac{du}{u^2-1}$$

which we could do partial fractions on. The answer is (after simplification) $\ln \frac{3}{2} \approx 0.4055$

Section 7.5 (Selected evens)

2. $\ln|1 - \cos(x)| + C$
4. $4\ln(2) - \frac{15}{16}$
6. $-\sin(\cos(x)) + C$
8. $\frac{1}{2}\sin^{-1}(x^2/\sqrt{3}) + C$
10. $\frac{\pi}{8} - \frac{1}{4}$
12. $\frac{-1}{3}\ln(5)$
14. $\frac{1}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}(x^2 + \frac{1}{2})) + C$
16. $3e^{\sqrt[3]{x}}(x^{2/3} - 2x^{1/3} + 2) + C$
18. $$2\sqrt{1+\ln|x|} + \ln\left(\frac{\sqrt{1+\ln|x|}+1}{\sqrt{1+\ln|x|}-1}\right) + C$$
20. $$\frac{1}{4}\left[(2x^2-1)\sin^{-1}(x) + x\sqrt{1-x^2}\right] + C$$
22. $\frac{52}{55}$
24. $\ln|x^3 - 2x - 8| + C$
26. $$-2\sqrt{\frac{t}{a}}\cos(\sqrt{at}) + \frac{2}{a}\sin(\sqrt{at}) + C$$
28. $$\frac{5}{8}\sin^{-1}\left(\frac{1}{\sqrt{5}}(2x-1)\right) + \frac{1}{4}(2x-1)\sqrt{1+x-x^2} + C$$
30. $\sqrt{2x-1} - 2\tan^{-1}(\frac{1}{2}\sqrt{2x-1}) + C$
32. $$\frac{1}{12}\ln|x-2| - \frac{1}{24}\ln(x^2+2x+4) - \frac{1}{4\sqrt{3}}\tan^{-1}(\frac{1}{\sqrt{3}}(x+1)) + C$$
34. $-\sin^{-1}(\frac{1}{3}\cos^2(x)) + C$
36. $-\frac{1}{2}\cos(x) - \frac{1}{14}\cos(7x) + C$
38. $\frac{5}{12}$

40.

$$\frac{1}{2}\ln\left|2y-1+\sqrt{4y^2-4y-3}\right|+C`$$

42. $\frac{1}{4}\tan(4x)-x+C$

44. $x-2\ln|e^x-1|+C$

46. $\frac{1}{4a^2}\ln|(x^2-a^2)/(x^2+a^2)|+C$

48.

$$\frac{1}{3}x^3\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\ln(x^2+1)+C$$

50. Let $A=\sqrt{4x+1}+1$ and $B=\sqrt{4x+1}-1$. Then the solution is:

$$2\ln(A)-\frac{2}{A}-2\ln(B)-\frac{2}{B}+C$$

Section 7.8

13. We can use u, du substitution to get that:

$$\int xe^{-x^2}\,dx=\frac{1}{2}\int e^{-u}\,du=\frac{-1}{2}e^{-x^2}$$

so

$$\int_{-\infty}^{\infty}xe^{-x^2}\,dx=$$

$$\lim_{t\to-\infty}\int_t^0 xe^{-x^2}\,dx+\lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx=$$

$$\lim_{t\to-\infty}\left(\frac{-1}{2}e^{-x^2}\right|_t^0+\lim_{t\to\infty}\left(\frac{-1}{2}e^{-x^2}\right|_0^t$$

which is:

$$\frac{-1}{2}+\frac{1}{2}=0$$

15. For $\int_0^\infty \frac{dx}{(x+2)(x+3)}$, there are no points of discontinuity. Just integrate using partial fractions and take the limit:

$$\int\frac{dx}{(x+2)(x+3)}=\ln|x+2|-\ln|x+3|=\ln\left|\frac{x+2}{x+3}\right|$$

so that

$$\lim_{t\to\infty}\int_0^t\frac{dx}{(x+2)(x+3)}=$$

$$\lim_{t\to\infty}\left(\ln\left|\frac{t+2}{t+3}\right|-\ln\left(\frac{2}{3}\right)\right)=\ln(1)-\ln(2/3)$$

$$=-\ln(2/3)$$

19. First, note that

$$\int_t^1 xe^{2x}\,dx=e^{2x}\left(\frac{1}{2}x-\frac{1}{4}\right)\Big|_t^1=$$

$$\frac{1}{4}e^2-e^{2t}\left(\frac{1}{2}t-\frac{1}{4}\right)$$

As $t\to-\infty$, by L'Hospital's rule:

$$\lim_{t\to\infty}\frac{\frac{1}{2}t-\frac{1}{4}}{e^{-2t}}=\lim_{t\to\infty}\frac{\frac{1}{2}}{e^{-2t}}=0$$

giving us a final answer of $e^2/4$

26. One way to integrate is to let $u=\ln(x)$, $du=\frac{1}{x}\,dx$, $e^u=x$, so that:

$$\int\frac{\ln(x)}{x^3}\,dx=\int ue^{2u}\,du=\frac{-1}{2}ue^{-2u}-\frac{1}{4}e^{-2u}=$$

$$\frac{1}{x^2}(-\frac{1}{2}\ln|x|-\frac{1}{4})$$

Now, we convert back to x and evaluate the limits:

$$\lim_{t\to\infty}\frac{1}{x^2}(-\frac{1}{2}\ln|x|-\frac{1}{4})\Big|_1^t$$

To compute this limit, we look at:

$$\lim_{t\to\infty}\frac{\ln(t)}{t^2}=\lim_{t\to\infty}\frac{1/t}{2t}=0$$

so overall, the limit is $\frac{1}{4}$.

33. Note that we already know that $\int_0^1\frac{1}{x^4}\,dx$ does not exist, so $\int_{-2}^3\frac{1}{x^4}\,dx$ does not exist, either.

36. First, we see that, by partial fractions:

$$\int_0^4\frac{1}{x^2+x-6}\,dx=\frac{-1}{5}\int_0^4\frac{1}{x+3}\,dx+\frac{1}{5}\int_0^4\frac{1}{x-2}\,dx$$

However, the last integral has a discontinuity at $x=2$, so that is the one we inspect:

$$\int_0^4\frac{1}{x-2}\,dx=\int_0^2\frac{1}{x-2}\,dx+\int_2^4\frac{1}{x-2}\,dx$$

But both of those integrals are divergent (either check directly or recall that $\int_0^1\frac{1}{x}\,dx$ was divergent).

40. First, integrate by parts to get that:

$$\int\frac{\ln(x)}{\sqrt{x}}\,dx=2\sqrt{x}\ln(x)-4\sqrt{x}$$

Now compute the limit:

$$\lim_{t\to 0^+} 2\sqrt{x}\ln(x) - 4\sqrt{x}\Big|_t^1$$

which means we need to examine:

$$\lim_{t\to 0^+} \frac{\ln(t)}{t^{-1/2}} = \lim_{t\to 0^+} \frac{1/t}{(-1/2)t^{-3/2}} = \lim_{t\to 0^+} -2\sqrt{t} = 0$$

so overall, the answer is -4.

42.

$$\int_0^\infty \mathrm{e}^{-x/2}\,dx = \lim_{t\to\infty} \left(-2\mathrm{e}^{-x/2}\Big|_0^t =\right.$$

To get a final anser of $2e$

51. First, you should look at the integral and guess that it probably converges, since the denominator has an exponential in it. Therefore, to use the Comparison Theorem, we look for a larger function that we know converges. To make the expression larger, we can make the denominator smaller:

$$\frac{1}{x+\mathrm{e}^{2x}} \le \frac{1}{\mathrm{e}^{2x}}$$

and we know that $\int_1^\infty \mathrm{e}^{-2x}\,dx$ converges. Note that even though the following statement is also true:

$$\frac{1}{x+\mathrm{e}^{2x}} \le \frac{1}{x}$$

but this says nothing about convergence or divergence of our original integral, since $\int_1^\infty (1/x)\,dx$ diverges.

58. Let $u = \ln(x)$, $du = \frac{1}{x}\,dx$, and change the bounds. Then:

$$\int_{\mathrm{e}}^\infty \frac{1}{x(\ln(x))^p}\,dx = \int_1^\infty u^{-p}\,du$$

so the integral converges if $p > 1$, diverges otherwise.