Integration by Substitution Solutions

1. $\int \frac{x}{\cos^2(x^2)}$ dx Let $u = x^2, du = 2x\,dx$ to get:

$$\frac{1}{2}\int \sec^2(u)\,du = \frac{1}{2}\tan(x^2) + C$$

2. $\int \frac{3}{\sqrt{5-9x^2}}$ dx First, some algebra (because we'd like the denominator to look like $\sqrt{1-a^2}$ for some a.

$$\frac{3}{\sqrt{5-9x^2}} = \frac{3}{\sqrt{5(1-\frac{9}{5}x^2)}} = \frac{3}{\sqrt{5}}\frac{1}{\sqrt{1-\left(\frac{3x}{\sqrt{5}}\right)^2}}$$

Now we see that, if $u = \frac{3}{\sqrt{5}}x$, the integral becomes:

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C = \sin^{-1}\left(\frac{3}{\sqrt{5}}\,x\right) + C$$

3. $\int \frac{6e^{1/x}}{x^2}$ dx Let $u = \frac{1}{x}, du = -\frac{1}{x^2}\,dx$. Performing the substitution, we get:

$$-\int 6e^u\,du = -6e^{1/x} + C$$

4. $\int \frac{e^x}{1+e^{2x}}$ dx Let $u = e^x, du = e^x\,dx$. Then:

$$\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C = \tan^{-1}(e^x) + C$$

5. $\int x^3\sqrt{x^4+11}$ dx Let $u = x^4+11, du = 4x^3\,dx$. Then:

$$\frac{1}{4}\int \sqrt{u}\,du = \frac{1}{4}\frac{2}{3}u^{3/2} + C = \frac{1}{6}(x^4+11)^{3/2} + C$$

6. $\int \frac{a^{\tan(t)}}{\cos^2(t)}$ dt Let $u = \tan(t), du = \sec^2(t)\,dt$. Noting that

$$\frac{1}{\cos^2(t)} = sec^2(t)$$

the integral becomes:

$$\int a^u\,du = \frac{1}{\ln(a)}a^u + C = \frac{1}{\ln(a)}a^{\tan(t)} + C$$

7. $\int \frac{x^2-x}{x+1}$ dx First, do long division to get:

$$\frac{x^2-x}{x+1} = x - 2 + \frac{2}{x+1}$$

Now,

$$\int x - 2 + \frac{2}{x+1}\,dx = \int x - 2\,dx + 2\int \frac{1}{x+1}\,dx$$

For the last integral, let $u = x+1, du = dx$. Then the antiderivative is:

$$\frac{1}{2}x^2 - 2x + 2\ln|x+1| + C$$

8. $\int_2^3 t\sqrt{t^2-4}$ dt Let $u = t^2-4$, so $du = 2t\,dt$ or $\frac{1}{2}\,du = t\,dt$ We now replace the integration limits in t by the integration limits in u:

If $t = 2$, then $u = 2^2 - 4 = 0$
If $t = 3$, then $u = 3^2 - 4 = 5$

So our integral becomes:

$$\int_2^3 t\sqrt{t^2-4}\,\mathrm{dt} = \frac{1}{2}\int_0^5 u^{1/2}\,du = \frac{1}{2}\frac{2}{3}u^{3/2}\Big|_0^5 = \frac{5\sqrt{5}}{3}$$

9. $\int \frac{e^{\sin(z)}}{\sec z}$ dz If we simplify a little first, we see that:

$$\int \frac{e^{\sin(z)}}{\sec z}\,\mathrm{dz} = \int e^{\sin(z)}\cos(z)\,\mathrm{dz}$$

Now substitute $u = \sin(z), du = \cos(z)\,dz$ and the solution is:

$$\int e^u\,du = e^u + C = e^{\sin(z)} + C$$

10. $\int_0^1 x10^{x^2}$ dx Let $u = x^2, \frac{1}{2}\,du = x\,dx$. We also need the new integration limits:

If $x = 0$, then $u = 0^2 = 0$
If $x = 1$, then $u = 1^2 = 1$

so that:

$$\int_0^1 x10^{x^2}\,\mathrm{dx} = \frac{1}{2}\int_0^1 10^u\,du = \frac{1}{2\ln(10)}10^u\Big|_0^1$$
$$= \frac{1}{2\ln(10)}(10-1) = \frac{9}{2\ln(10)}$$

11. $\int \frac{t^2\cos(t^3-2)}{\sin^2(t^3-2)}$ dt A DOUBLE WHAMMY!

First, let $v = t^3-2, dv = 3t^2\,dt$, so that:

$$\int \frac{t^2\cos(t^3-2)}{\sin^2(t^3-2)}\,\mathrm{dt} = \frac{1}{3}\int \frac{\cos(v)}{\sin^2(v)}\,dv$$

Now, let $u = \sin(v)$, so $du = \cos(v)\,dv$. Then:

$$\frac{1}{3}\int \frac{\cos(v)}{\sin^2(v)}\,dv = \frac{1}{3}\int u^{-2}\,du$$

Backsubstituting:

$$-u^{-1} + C = -(\sin(v))^{-1} + C = -\csc(t^3-2) + C$$

12. $\int \frac{x}{x^4+1}$ dx Let $u = x^2, \frac{1}{2}\,du = x\,dx$. Then:

$$\frac{1}{2}\int \frac{1}{u^2}\,du = \frac{1}{2}\tan^{-1}(x^2) + C$$

13. $\int \frac{\cos(\ln(4x^2))}{x}$ dx Let $u = \ln(4x^2)$. Then:

$$\frac{du}{dx} = \frac{1}{4x^2}8x = \frac{8x}{4x^2} = \frac{2}{x}$$

so that $\frac{1}{2}\,du = \frac{1}{x}\,dx$, from which we get:

$$\int \cos(u)\,du = \sin(u) + C = \sin(\ln(4x^2)) + C$$

14. $\int \frac{\tan(z)}{\cos(z)}$ dz Rewriting the integrand, we get:

$$\int \frac{\tan(z)}{\cos(z)}\,\mathrm{dz} = \int \frac{\sin(z)}{\cos^2(z)}\,\mathrm{dz}$$

Now we let $u = \cos(z)$, and $-du = \sin(z)\,dz$. Now,

$$-\int \frac{1}{u^2}\,du = u^{-1} + C = \frac{1}{\cos(z)} + C = \sec z + C$$

15. $\displaystyle\int \frac{\mathrm{e}^{\tan^{-1}(2t)}}{1+4t^2}\,\mathrm{dt}$ Let $u = \tan^{-1}(2t)$. Then

$$\frac{1}{2}du = \frac{1}{1+4t^2}\;dt$$

The integral becomes:

$$\int \mathrm{e}^u\;du = \mathrm{e}^u + C = \mathrm{e}^{\tan^{-1}(2t)} + C$$

16. $\displaystyle\int_0^1 x^2(1+2x^3)^5\,\mathrm{dx}$ Let $u = 1+2x^3$. Then $\frac{1}{6}du = x^2\;dx$. We also compute the new limits of integration:

If $x = 0$, then $u = 1 + 2\cdot 0^3 = 1$
If $x = 1$, then $u = 1 + 2\cdot 1^3 = 3$

Now we have:

$$\frac{1}{6}\int_1^3 u^5\;du = \frac{1}{6}\frac{1}{6}u^6\Big|_1^3 = \frac{1}{36}\left(3^6-1\right)$$

17. $\displaystyle\int_0^3 \frac{\mathrm{dx}}{2x+3}$ Let $u = 2x+3$, then $\frac{1}{2};du = dx$. Replacing limits:

If $x = 0$, then $u = 3$
If $x = 1$, then $u = 5$

$$\frac{1}{2}\int_3^5 \frac{1}{u}\;du = \frac{1}{2}\ln(u)\Big|_3^5 = \frac{1}{2}\left(\ln(5) - \ln(3)\right)$$

$$= \frac{1}{2}\ln\left(\frac{5}{3}\right) = \ln\left(\sqrt{\frac{5}{3}}\right)$$

18. $\displaystyle\int_{\mathrm{e}}^{\mathrm{e}^4} \frac{\mathrm{dx}}{x\sqrt{\ln(x)}}$ Let $u = \ln(x)$. Then $du = \frac{1}{x}\;dx$, and the limits of integration: $x = e, u = 1$, $x = e^4, u = 4$:

$$\int_1^4 u^{1/2}\;du = \frac{2}{3}u^{3/2}\Big|_1^4$$

$$= \frac{2}{3}\left(4^{3/2}-1\right) = \frac{14}{3}$$

19. $\displaystyle\int_0^1 \cos(\pi t)\;\mathrm{dt}$ Let $u = \pi t$, $\frac{1}{\pi}\;du = dt$, so the integral becomes (remember to change the limits):

$$\frac{1}{\pi}\int_0^\pi \cos(u)\;du = \frac{1}{\pi}\sin(u)\Big|_0^\pi = 0$$

20. $\displaystyle\int_1^4 \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\;\mathrm{dx}$ Let $u = 1+\frac{1}{x}$, so $-\;du = \frac{1}{x^2}\;dx$, so the integral becomes:

$$-\int_2^{\frac{5}{4}} \sqrt{u}\;du = \int_{5/4}^2 u^{1/2}\;du = \frac{2}{3}u^{3/2}\Big|_{5/4}^2$$

$$= \frac{2}{3}\left(2^{3/2} - \left(\frac{5}{4}\right)^{3/2}\right)$$