Extra Practice: #10, 57, 74 in 14.3

10. Note that the origin is one up from the bottom of the figure, and 3 units from the left of the figure. That puts the point $(2,1)$ very close to the contour $f(x,y)=10$.

    Consider how far you need to go in each direction (up/down, left/right) to get to the next contour.

    - Right: Go about 0.6 units to get to 12.
    - Left: Go about 0.8 units to get to 8.
    - Up: Go about 0.8 units to get to 8.
    - Down: Go about 1 unit to get to 12.

    Therefore,

$$f_x(2,1) \approx \frac{1}{2}\left(\frac{2}{0.8}+\frac{2}{0.6}\right) \approx 2.9$$
$$f_y(2,1) \approx \frac{1}{2}\left(\frac{-2}{1}+\frac{-2}{0.8}\right) \approx -2.3$$

57. This one is "algebra-heavy":

$$z = \arctan\left(\frac{x+y}{1-xy}\right)$$

Differentiating both sides with respect to x:

$$z_x = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1)(1-xy)+(x+y)(-y)}{(1-xy)^2} =$$

$$\frac{(1-xy)^2}{(1-xy)^2+(x+y)^2}\frac{1+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2+(x+y)^2} =$$
$$\frac{1+y^2}{1+x^2+y^2+x^2y^2} = \frac{1+y^2}{(1+x^2)(1+y^2)} = \frac{1}{1+x^2}$$

If you didn't get this last step, leaving your answer one step before the finish here would have been fine.

For z_y, we have a completely symmetric problem:

$$z_y = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1)(1-xy)+(x+y)(-x)}{(1-xy)^2} =$$

$$\frac{(1-xy)^2}{(1-xy)^2+(x+y)^2}\frac{1+x^2}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2+(x+y)^2} = \cdots = \frac{1}{1+y^2}$$

74. We are considering the first and second partial derivatives at the point P.

(a) For f_x, we look for how f changes as x changes to the right and left. Moving right, we go "downhill", so $f_x < 0$.

(b) For f_y, we look for how f changes as we move up/down. By moving up (increasing y), we see that f is increasing. Therefore, $f_y > 0$.

(c) For f_{xx}, we want to see how f_x is changing as we move left/right. We start with $f_x < 0$, and moving right, it takes more space to get to the next contour- so the surface is becomes less steep. Therefore, f_x is increasing, so $f_{xx} > 0$.

(d) For f_{xy}, we start with $f_x < 0$, and we notice that the contours are getting closer as we go up (the distance down to $z = 4$ is quite a bit more than the distance up to $z = 8$). Therefore, f_x becomes more negative (decreasing), so $f_{xy} < 0$.

(e) For f_{yy}, we start with $f_y > 0$ and see how f_y changes by moving up/down. Again, the level curves get closer as we move up, meaning that f_y is getting more positive. Therefore, $f_{yy} > 0$.