Final Exam Pack C

1. Short Answer/True or False. You may assume that all vectors are in $\mathbb{R}^3$.
   (a) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.
   TRUE. The cross product is orthogonal to each of the vectors.
   (b) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$
   TRUE. This formula is for the cross product, and we had a similar formula (involving the cosine) for the dot product.
   (c) Given $\mathbf{r}(t)$, then $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ and $\mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)|$.
   TRUE. We showed that if a curve has a fixed magnitude, then the derivative is orthogonal to the curve. (The unit tangent vector always has unit length).

2. If $\mathbf{a} \cdot \mathbf{b} = \sqrt{3}$ and $\mathbf{a} \times \mathbf{b} = \langle 1, 2, 2 \rangle$, find the angle between $\mathbf{a}, \mathbf{b}$.
   SOLUTION: There are several ways to get at this one, here's one way:
   $$\sqrt{3} = \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$$
   and
   $$3 = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$$
   so that
   $$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$
   (From the 30-60-90 triangle).

3. Find the limit, if it exists: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4 + y^4}$
   SOLUTION: Along the $x-$ or $y-$ axis, the limit is zero. Along $y = x$, we get something different:
   $$\lim_{x\to 0} \frac{6x^4}{3x^4} = 2$$
   Therefore, the limit does not exist overall.

4. Find the distance from $(1, 2, 3)$ to the plane $x + 2y - z + 3 = 0$.
   SOLUTION:
   $$\frac{|(1) + 2(2) - (3) + 3|}{\sqrt{1 + 4 + 1}} = \frac{5}{\sqrt{6}}$$

5. Find the point at which the line intersects the plane:
   $$x = 3 - t, \quad y = 2 + t, \quad z = 5t, \qquad x - y + 2z = 9$$
   SOLUTION: Substitute into the equation for the plane, solve for t. In this case, $t = 1$, so the point is $(2, 3, 5)$.

6. Find the equation of the plane that passes through $(1, 2, 3)$ and contains the line $x = 3t, y = 1 + t, z = 2 - t$.
   SOLUTION: Using a point and a line, we can get two directions and then a normal vector. One direction from the line itself is $\langle 3, 1, -1 \rangle$. For another direction, get a point on the line, like $(0, 1, 2)$, then form a vector using the other point $(1, 2, 3)$, which will be $\langle 1, 1, 1 \rangle$. Take the cross product, and get
   $$\mathbf{n} = \langle -2, 4, -2 \rangle$$
   Therefore, the equation of the plane is:
   $$-2(x - 1) + 4(y - 2) - 2(z - 3) = 0$$

7. Verify the approximation: $\frac{2x+3}{4y+1} \approx 3 + 2x - 12y$

   SOLUTION: The idea is that this should be a linear approximation, at $(0, 0)$. In this case, if

$$F(x,y) = \frac{2x+3}{4y+1} \qquad \Rightarrow \qquad F_x = \frac{2}{4y+1} \qquad F_y = \frac{4(2x+3)}{(4y+1)^2}$$

   We see that $F_x(0,0) = 2$ and $F_y(0,0) = -12$. Therefore, the tangent plane is:

$$F(0,0) + F_x(0,0)(x-0) + F_y(0,0)(y-0) = 3 + 2x - 12y$$

8. Find the rate of change of f at the point $(0, 2)$ in the direction of $(1, 1)$, if $f(x, y) = y\mathrm{e}^{-x}$.

   SOLUTION: Notice that $(1, 1)$ is another point, so that the direction of $\mathbf{u}$ is given by $\langle 1-0, 1-2\rangle = \langle 1, -1\rangle$. Normalize for the directional derivative:

$$D_u f = \nabla f(0,2) \cdot \frac{1}{\sqrt{2}}\langle 1, -1\rangle = -\frac{3}{\sqrt{2}}$$

9. Find and classify the critical points of $f(x, y) = xy - 2x - 2y - x^2 - y^2$.

   SOLUTION: This is the test for local extrema. Find the critical points first:

$$f_x = -2x + y - 2 \qquad f_y = x - 2y - 2$$

   Setting these to zero, solve for x, y. We get $x = -2, y = -2$. Now we check the "second derivatives".

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 3 > 0 \text{ Also, } f_{xx} < 0, \quad \Rightarrow \quad \text{Local Max}$$

10. Use Lagrange Multipliers to find the maximum of $f(x, y) = 3x + y$ with the constraint $x^2 + y^2 = 10$.

    SOLUTION: Putting together the appropriate equations, we can solve the equations for x, y and substitute into the third:

$$\begin{array}{rl} 3 & = \lambda 2x \\ 1 & = \lambda 2y \\ x^2+y^2 & = 10 \end{array} \quad \Rightarrow \quad \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 10 \quad \Rightarrow \quad \lambda = \pm\frac{1}{2}$$

    If $\lambda = \frac{1}{2}$, we get $x = 3, y = 1$ and a maximum of 10. If $\lambda = -\frac{1}{2}$, we get $x = -3$, $y = -1$ and a minimum of -10.

11. Evaluate $\iint_D y^2 \mathrm{e}^{xy}\, dA$, where D is bounded by $y = x, y = 4$ and $x = 0$. Be sure to choose the order of integration that will be easier!

    SOLUTION: Draw the region, it is the upper triangle. Integrate in x first,

$$\int_0^4 \int_0^y y^2 \mathrm{e}^{xy}\, dx\, dy = \frac{1}{2}\mathrm{e}^{16} - \frac{17}{2}$$

12. Let solid E be the solid bounded by paraboloid $y = x^2 + z^2$ and the plane $y = 0$.

    TYPO: The plane should be $y = 4$, not $y = 0$

    (a) Write two triple integrals that will give the volume, one where the order of integration is $dz\, dy\, dx$, and one where the order is $dy\, dz\, dx$. (Do not evaluate these).
    SOLUTION:

$$\int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} dz\, dy\, dx \qquad \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} dy\, dz\, dx$$

(b) If we have a vector field $\mathbf{F} = \langle y, z, -x \rangle$, then write an integral that represents the flux of $\mathbf{F}$ across the surface of the paraboloid (you can ignore the plane $y = 4$). (Again, just write the integral, do not evaluate).

SOLUTION: Your integral will depend on how you want to represent the surface. Here are two choices:

- Two surfaces, $z = \pm\sqrt{y - x^2}$ (positive side on the left, negative on the right)

$$\mathbf{r}_x \times \mathbf{r}_y = \left\langle \frac{x}{\sqrt{y-x^2}}, -\frac{1}{2\sqrt{y-x^2}}, 1 \right\rangle \qquad \mathbf{r}_x \times \mathbf{r}_y = \left\langle \frac{x}{\sqrt{y-x^2}}, -\frac{1}{2\sqrt{y-x^2}}, -1 \right\rangle$$

  Now we integrate the dot products with $\mathbf{F}$:

$$\int_{-2}^{2} \int_{x^2}^{4} \frac{2xy - \sqrt{y-x^2}}{2\sqrt{y-x^2}} - x\, dy\, dx + \int_{-2}^{2} \int_{x^2}^{4} \frac{2xy + \sqrt{y-x^2}}{2\sqrt{y-x^2}} + x\, dy\, dx$$

- A nicer choice might be to parameterize: $\mathbf{r}(u, v) = \langle u\cos(v), u^2, u\sin(v) \rangle$.
  If we compute the cross product,

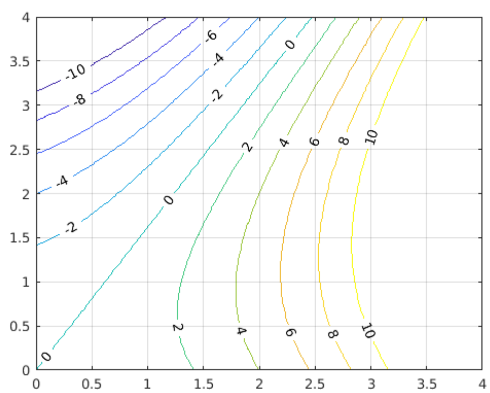
$$\mathbf{r}_u \times \mathbf{r}_v = \langle 2u^2\cos(v), -u, 2u^2\sin(v) \rangle$$

  Simplifying $\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)$, we would integrate:

$$\int_0^{2\pi} \int_0^2 2u^4\cos(v) - u^2\sin(v) - 2u^3\sin(v)\cos(v)\, du\, dv$$

(And FYI, the flux ends up being 0).

13. Given below is the plot of some contours of $f(x, y)$. At the point $(2, 2)$, estimate the sign (positive, negative, or zero) of f_x, f_y, f_{yy} and f_{yx}.



- $f_x > 0$, $f_y < 0$
- f_{yy} : The level curves get closer as you move up, so the curve get steeper in a negative way- $f_{yy} < 0$.
- f_{yx} is a little trickier- It might be easier to estimate f_{xy}, and in this case both are positive.

14. Set up, but do not evaluate, the line integral $\int_C xy\, dy$, if the curve C is the bottom half of the unit circle going counterclockwise (CCW).

SOLUTION: We can parameterize the curve as $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ with $\pi \le t \le 2\pi$. Substitution:

$$\int_{\pi}^{2\pi} \cos^2(t)\sin(t)\, dt$$

15. Evaluate $\int_C y^3\, dx - x^3\, dy$ if C is the circle $x^2 + y^2 = 4$.

SOLUTION: Assuming CCW rotation, we can use Green's Theorem, which is easiest to express in polar coordinates:

$$\iint -3(x^2+y^2)\,dA = -3\int_0^{2\pi}\int_0^2 r^2\,r\,dr\,d\theta$$

16. Let C be a simple closed smooth curve that lies in the plane $x+y+z=1$. Show that the line integral below depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

$$\int_C z\,dx - 2x\,dy + 3y\,dz$$

SOLUTION: The integral is in the form $\int_C \mathbf{F}\cdot d\mathbf{r}$, and the curl of $\mathbf{F}$ is $\langle 3, 1, -2\rangle$. The surface normal is $\langle 1, 1, 1\rangle$, so, using Stokes' Theorem, we get:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \mathrm{curl}(\mathbf{F})\cdot d\mathbf{S} = \iint_S \mathrm{curl}(\mathbf{F})\cdot \frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{g_x^2+g_y^2+1}}dS = \frac{2}{\sqrt{3}}\iint_S dS$$

This is $2/\sqrt{3}$ times the surface area bounded by the curve C.

17. Verify the divergence theorem, for the vector field $\mathbf{F} = \langle x, y, z\rangle$ where E is the unit ball $x^2+y^2+z^2 \leq 1$.

SOLUTION: I won't ask you to compute the flux of a vector field across the surface of a sphere- To do that requires a lot of work, since we use spherical coordinates to parameterize the surface. (To see what I mean, look at Example 4, Section 16.7, page 1117) We would typically do that via Maple.

This question should have been to use the Divergence Theorem to compute the flux of the given vector field across the surface, because that is very straightforward- The divergence is 3, so we get:

$$3\times \text{ vol of unit sphere} = 3\frac{4}{3}\pi = 4\pi$$