16.6: Parametric Surfaces

We'll recall that initially, we defined **curves in the plane** as $y = f(x)$, but that does not cover all of the ways we might actually draw a curve in the plane, and it does not extend to three dimensions.

Rather, we defined a **curve** parametrically, so that

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

where x, y, z are functions of another variable, t.

Similarly, it is awkward to denote all possible surfaces by $z = f(x, y)$. Therefore, we will define a **parametric surface** to be:

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

Example 1 (Prototype, or Template)

If we are given $z = f(x, y)$, how might we convert the function to u, v?

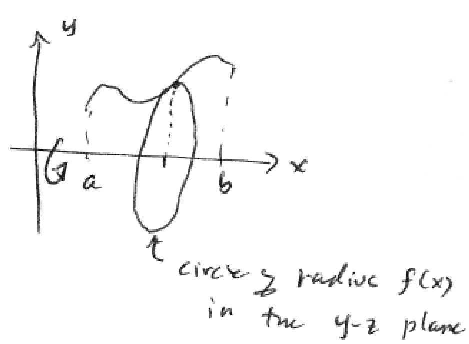
$$\mathbf{r}(u, v) = \langle u, v, f(u, v) \rangle$$

In this case, u, v take the place of x, y, respectively. Notice that it is easy to write either x as a function of y, z or y as a function of x and z using this technique:

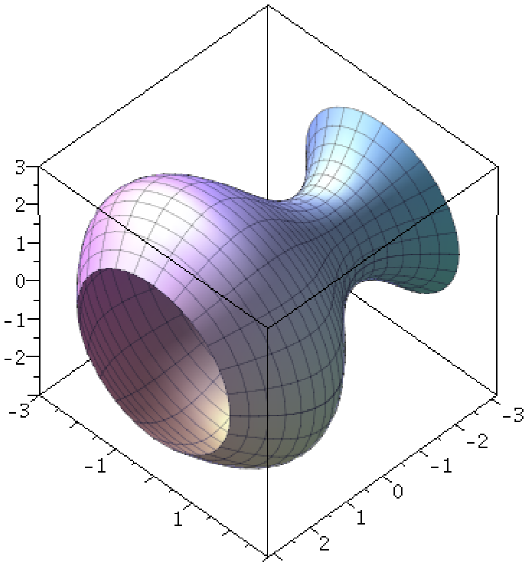
$$\mathbf{r}(u, v) = \langle f(u, v), u, v \rangle \qquad \text{or} \qquad \mathbf{r}(u, v) = \langle u, f(u, v), v \rangle$$

Example 2 (Template)

We can write down the surface equations for a surface of rotation. For example, suppose we take $f(x) = \sin(x) + 2$, $-\pi \le x \le \pi$, and rotate that around the $x-$axis.

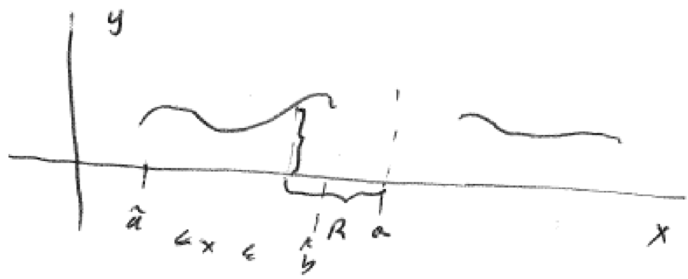


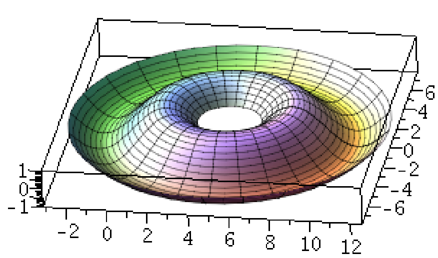
If we freeze the value of x, we get a circle in the yz plane with a radius of $f(x)$. Therefore, the equations would be:



$$
\begin{aligned}
f(x) &= \sin(x) + 2 \\
\\
x &= x \\
y &= f(x)\cos(\theta) \\
z &= f(x)\sin(\theta)
\end{aligned}
$$

Now, let's try rotating the curve about the line $x = 3\pi/2$. In that case, with $-\pi \leq x \leq \pi$, the radius of the circle will be $R = 3\pi/2 - x$ in the xz plane. The center of the circle will be $x = 3\pi/2$ and $z = 0$, and finally the height will be $f(x)$.





$$
\begin{aligned}
f(x) &= \sin(x) + 2 \\
\\
x &= ((3\pi/2) - x)\cos(\theta) + 3\pi/2 \\
y &= f(x) \\
z &= ((3\pi/2) - x)\sin(\theta)
\end{aligned}
$$

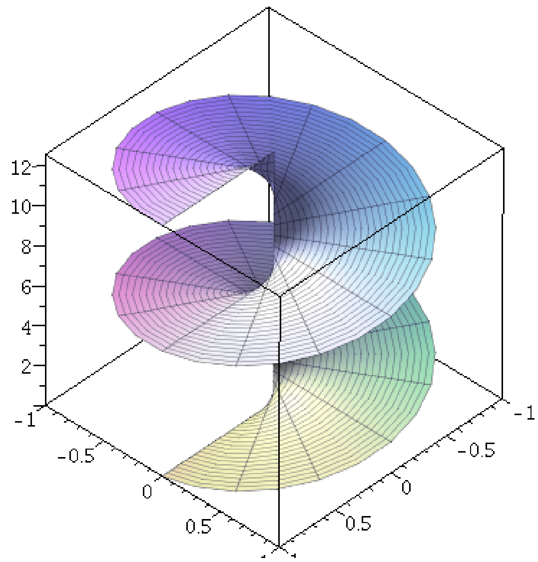
Understanding a parametric surface

To understand how a parametric surface is put together, we will typically have a rectangular domain: $a \le u \le b$ and $c \le v \le d$, and we can see how lines in the domain are plotted to the surface.

Example

The equations to explore are the following:

$$\begin{aligned} x &= v\cos(u) \\ y &= v\sin(u) \\ z &= u \end{aligned}$$



with $0 \le u \le 4\pi$ and $0 \le v \le 1$. This creates a rectangle in the (u, v) plane that we can map to 3d as shown below. The easiest way to see what's happening is to freeze one variable and look to see how the other variable is mapped. For example, suppose we freeze $v = 1$ and let $0 \le u \le 4\pi$. The equations become the equations of a helix that wraps around the $z-$ axis twice as z increases to 4π.

If we freeze $v = 1/2$, we get another helix, but with a tighter radius.

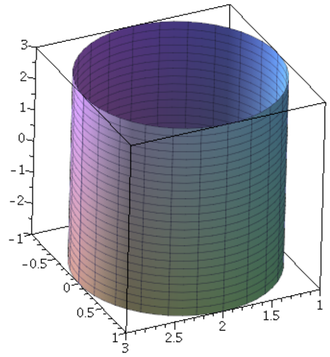
Alternatively, if we freeze u, that freezes z at height u, and we get a ray emanating from the z axis as v increases.

Other sources of parametric functions

We might use parametric equations to define surfaces in other coordinate systems.

For example, suppose I want to define a cylinder whose xy-plane cross section is given by: $(x-2)^2 + y^2 = 1$. How would we do that?

$$
\begin{aligned}
x &= \cos(\theta) + 2 \\
y &= \sin(\theta) \\
z &= z
\end{aligned}
$$



Here is the Maple code used for the surfaces shown:

```
with(plots):
f:=sin(x)+2;
plot3d([x,f*cos(u), f*sin(u)],x=-Pi..Pi,u=0..2*Pi);
A:=3*Pi/2;
plot3d([(A-x)*cos(u)+A, (A-x)*sin(u),sin(x)],x=-Pi..Pi,u=0..2*Pi);
plot3d([v*cos(u), v*sin(u), u], u=0..4*Pi, v=0..1);
plot3d([cos(theta)+2,sin(theta),z],theta=0..2*Pi,z=-3..3);
```