Review: Integration by Parts (IBP)

We typically see integration by parts as:

$$\int u\,dv = uv - \int v\,du$$

And this work well for problems where we only need to do IBP once. For example,

$$\int \ln(x)\,dx \quad \Rightarrow \quad \begin{array}{cc} u = \ln(x) & dv = dx \\ du = 1/x\,dx & v = x \end{array} \quad \Rightarrow \quad x\ln(x) - \int \frac{x}{x}\,dx = x\ln(x) - x + C$$

As an alternate form, we could write:

$$\int f(x)\,g'(x)\,dx \quad \Rightarrow \quad \begin{array}{ccc} \text{sign} & \text{Diff} & \text{Int} \\ \hline + & f(x) & g'(x) \\ - & f'(x) & g(x) \end{array} \quad \Rightarrow \quad f(x)g(x) - \int g(x)f'(x)\,dx$$

In fact, we can go farther:

$$\int f(x)\,g''(x)\,dx \quad \Rightarrow \quad \begin{array}{ccc} \text{sign} & \text{Diff} & \text{Int} \\ \hline + & f(x) & g''(x) \\ - & f'(x) & g'(x) \\ + & f''(x) & g(x) \end{array} \quad \Rightarrow \quad f(x)g'(x) - f'(x)g(x) + \int g(x)f''(x)\,dx$$

And further still:

$$\int f(x)\,g'''(x)\,dx \Rightarrow \begin{array}{ccc} \text{sign} & \text{Diff} & \text{Int} \\ \hline + & f(x) & g'''(x) \\ - & f'(x) & g''(x) \\ + & f''(x) & g'(x) \\ - & f'''(x) & g(x) \end{array} \Rightarrow f(x)g''(x) - f'(x)g'(x) + f''(x)g(x) - \int g(x)f'''(x)\,dx$$

This technique works very well when the derivatives of f are 0 (like for polynomials):

$$\int x^2 \mathrm{e}^{3x}\,dx$$

For this, we take $f(x) = x^2$:

$$\begin{array}{ccc} \text{sign} & \text{Diff} & \text{Int} \\ \hline + & x^2 & \mathrm{e}^{3x} \\ - & 2x & \frac{1}{3}\mathrm{e}^{3x} \\ + & 2 & \frac{1}{9}\mathrm{e}^{3x} \\ - & 0 & \frac{1}{27}\mathrm{e}^{3x} \end{array} \Rightarrow \mathrm{e}^{3x}\left(\frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right) - \frac{1}{27}\int 0\cdot \mathrm{e}^{3x}\,dx$$

That last integral is just zero (it's included so you see the pattern).

Below, we'll work out a couple more examples, then have some sample problems with solutions.

Worked Examples

1. $t^2\sin(2t)\,dt$

   Just as in the last example, we take $f(t) = t^2$ so that the derivative is eventually 0.

$$\begin{array}{ccr} \text{sign} & \text{Diff} & \text{Int} \\ \hline + & t^2 & \sin(2t) \\ - & 2t & -\frac{1}{2}\cos(2t) \\ + & 2 & -\frac{1}{4}\sin(2t) \\ - & 0 & \frac{1}{8}\cos(2t) \end{array} \Rightarrow -\frac{t^2}{2}\cos(2t) + \frac{t}{2}\sin(2t) + \frac{1}{4}\cos(2t) + C$$

2. $\displaystyle\int e^{2t}\cos(3t)\,dt$

For this problem, we compute IBP twice in order to get the same integral to appear on both sides of the equation. Let's start and see what that means:

sign	Diff	Int
$+$	e^{2t}	$\cos(3t)$
$-$	$2e^{2t}$	$\frac{1}{3}\sin(3t)$
$+$	$4e^{2t}$	$-\frac{1}{9}\cos(3t)$

We interpret this to mean:

$$\int e^{2t}\cos(3t)\,dt = e^{2t}\left(\frac{1}{3}\sin(3t)+\frac{2}{9}\cos(3t)\right)-\frac{4}{9}\int e^{2t}\cos(3t)\,dt$$

Add $\frac{4}{9}\int e^{2t}\cos(3t)\,dt$ to both sides, and solve for the integral:

$$\frac{13}{9}\int e^{2t}\cos(3t)\,dt = e^{2t}\left(\frac{1}{3}\sin(3t)+\frac{2}{9}\cos(3t)\right)$$

$$\int e^{2t}\cos(3t)\,dt = e^{2t}\left(\frac{3}{13}\sin(3t)+\frac{2}{13}\cos(3t)\right)+C$$

3. $\int x\ln(x)\,dx$

We can't really use the shortcuts here, since the antiderivatives of $\ln(x)$ are themselves IBP problems. In that case, we'll put $f(x)=\ln(x)$ instead:

sign	Diff	Int
$+$	$\ln(x)$	x
$-$	$1/x$	$\frac{1}{2}x^2$

$$\Rightarrow\quad \frac{1}{2}x^2\ln(x)-\frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln(x)-\frac{1}{4}x^2+C$$

Practice Problems

1. $\displaystyle\int \sqrt{x}\ln(x)\,dx$
2. $\displaystyle\int x^2\cos(3x)\,dx$
3. $\displaystyle\int t^3 e^{-2t}\,dt$
4. $\displaystyle\int e^{-2t}\sin(2t)\,dt$
5. $\displaystyle\int \tan^{-1}(1/t)\,dt$
6. $\displaystyle\int t\sin(t)\,dt$

Solutions to the Practice (online)

1. $\displaystyle\int \sqrt{x}\ln(x)\,dx$

sign	Diff	Int
$+$	$\ln(x)$	$x^{1/2}$
$-$	$1/x$	$\frac{2}{3}x^{3/2}$

$$\Rightarrow\quad \frac{2}{3}x^{3/2}\ln(x)-\frac{2}{3}\int x^{1/2}\,dx = \frac{2}{3}x^{3/2}\ln(x)-\frac{4}{9}x^{3/2}+C$$

2. $\displaystyle\int x^2\cos(3x)\,dx$

sign	Diff	Int
$+$	x^2	$\cos(3x)$
$-$	$2x$	$\frac{1}{3}\sin(3x)$
$+$	2	$-\frac{1}{9}\cos(3x)$
$-$	0	$-\frac{1}{27}\sin(3x)$

$$\Rightarrow\quad \frac{1}{3}x^2\sin(3x)+\frac{2x}{9}\cos(3x)-\frac{2}{27}\sin(3x)+C$$

3. $\displaystyle\int t^3 e^{-2t}\,dt$

sign	Diff	Int
$+$	t^3	e^{-2t}
$-$	$3t^2$	$-\frac{1}{2}e^{-2t}$
$+$	$6t$	$\frac{1}{4}e^{-2t}$
$-$	6	$-\frac{1}{8}e^{-2t}$
$+$	0	$\frac{1}{16}e^{-2t}$

$$\Rightarrow\ e^{-2t}\left(-\frac{1}{2}t^2-\frac{3}{4}t^2-\frac{3}{4}t-\frac{3}{8}\right)+C$$

4. $\displaystyle\int e^{-2t}\sin(2t)\,dt$

In the example, we had put the exponential in the middle, but we could have switched that with the sine, which we'll do below (either way is correct):

sign	Diff	Int
$+$	$\sin(2t)$	e^{-2t}
$-$	$2\cos(2t)$	$-\frac{1}{2}e^{-2t}$
$+$	$-4\sin(2t)$	$\frac{1}{4}e^{-2t}$

Therefore,

$$\int e^{-2t}\sin(2t)\,dt = e^{-2t}\left(-\frac{1}{2}\sin(2t)-\frac{1}{2}\cos(2t)\right)-\int e^{-2t}\sin(2t)\,dt$$

so that (add the integral to both sides, divide both sides by 2)

$$\int e^{-2t}\sin(2t)\,dt = -\frac{1}{4}e^{-2t}\left(\sin(2t)+\cos(2t)\right)+C$$

5. $\displaystyle\int \tan^{-1}(1/t)\,dt$

In the old style, we let $u=\tan^{-1}(1/t)$. Differentiate to get du:

$$du=\frac{1}{1+(1/t)^2}\cdot\frac{-1}{t^2}=\frac{-1}{1+t^2}$$

Further, if $dv=dt$, then $v=t$, and we get:

$$t\arctan(1/t)-\int\frac{-t}{1+t^2}\,dt=t\arctan(t)+\frac{1}{2}\ln(t^2+1)+C$$

6. $\displaystyle\int t\sin(t)\,dt$

$+$	t	$\sin(t)$
$-$	1	$-\cos(t)$
$+$	0	$-\sin(t)$

$$\Rightarrow\quad -t\cos(t)+\sin(t)+C$$