Integrating Powers of Sine and Cosine

There are three really important identities that you really ought to commit to memory, since so many other identities stem from these:

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

Using the third identity, we get the **half angle formulas**:

$$\cos(2\theta) = \cos^2(\theta) - (1 - \cos^2(\theta)) = 2\cos^2(\theta) - 1 \quad \Leftrightarrow \quad \cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$\cos(2\theta) = (1 - \sin^2(\theta)) - \sin^2(\theta) = 1 - 2\sin^2(\theta) \quad \Leftrightarrow \quad \sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

You should know these formulas.

Integrate $\int \sin^m(x)\cos^n(x)\,dx$ when m or n is odd (or both)

The idea: Take the term with the odd power, and reserve one to go with dx in preparation for u, du substitution. Here are a couple of examples:

- $\int \sin^m(x)\cos(x)\,dx$ (In this case, m can be even or odd, and $n = 1$)

  SOLUTION: Let $u = \sin(x)$ so that $du = \cos(x)\,dx$, and the integral becomes (for $m > 0$)

$$\int u^m\,du = \frac{1}{m+1}\sin^{m+1}(x) + C$$

- $\int \sin^3(x)\cos^2(x)\,dx$

  SOLUTION: Reserve one $\sin(x)$ for the substitution, then write everything else in terms of $\cos(x)$. We will let $u = \cos(x)$ so $du = -\sin(x)\,dx$:

$$\int \sin^2(x)\cos^2(x)\,(\sin(x)\,dx) = \int (1 - \cos^2(x))\cos^2(x)\,(\sin(x)\,dx) =$$

$$-\int (1 - u^2)u^2\,du = \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C$$

- Similarly: $\int \sin^3(x)\,dx = \int \sin^2(x)[\sin(x)\,dx] = \int (1 - \cos^2(x))[\sin(x)\,dx]$

Integrating when both powers are even

In this case, we integrate $\int \sin^m(x)\cos^n(x)\,dx$, when both m and n are even.

The idea: Use the half angle formulas to reduce the exponents, hopefully to something we can integrate. Often, this step is taken multiple times.

- $\int \sin^2(x)\,dx = \int \frac{1}{2}(1-\cos(2x))\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
- $\int \sin^2(x)\cos^2(x)\,dx$

  Method 1: Use half angle formulas:

  $$\sin^2(x)\cos^2(x) = \frac{1-\cos(2x)}{2}\frac{1+\cos(2x)}{2} = \frac{1}{4}(1-\cos^2(2x))$$

  Similarly, $\cos^2(2x) = \frac{1+\cos(4x)}{2}$, so this further reduces to:

  $$\sin^2(x)\cos^2(x) = \frac{1}{4} - \frac{1}{4}\cdot\frac{1+\cos(4x)}{2} = \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4x) = \frac{1}{8} - \frac{1}{8}\cos(4x)$$

  And this is easy to integrate: $\frac{1}{8}x - \frac{1}{32}\sin(4x) + C$

  Method 2: Use the identity for $\sin(2x)$:

  $$\sin^2(x)\cos^2(x) = (\sin(x)\cos(x))^2 = \left(\frac{\sin(2x)}{2}\right)^2 = \frac{1}{4}\sin^2(2x) = \frac{1}{4}\left(\frac{1-\cos(4x)}{2}\right)$$

Exercises

1. $\int \sin^2(x)\cos^3(x)\,dx$
2. $\int \sin^7(x)\cos^5(x)\,dx$
3. $\int \cos^2(x)\,dx$
4. $\int \sin^4(x)\,dx$
5. $\int x\sin^2(x)\,dx$
6. $\int x\sin^3(x)\,dx$

Solutions

1. $\int \sin^2(x)\cos^3(x)\,dx$

   SOLUTION: Using the odd power, we reserve $\cos(x)$ with dx to use as du. Rewrite what remains in terms of sines (using the Pythagorean Identity):

   $$\int \sin^2(x)\cos^2(x)\,[\cos(x)\,dx] = \int \sin^2(x)(1-\sin^2(x))\,[\cos(x)\,dx]$$

   Now, take $u = \sin(x)$ so $du = \cos(x)\,dx$, and substitute:

   $$\int u^2 - u^4\,du = \frac{1}{3}u^3 - \frac{1}{5}u^5 = \frac{1}{3}\sin^3(\theta) - \frac{1}{5}\sin^5(\theta) + C$$

2. $\int \sin^7(x)\cos^5(x)\,dx$

   SOLUTION: We can choose to reserve either a sine or a cosine, since both powers are odd. Choosing a $\cos(x)$, we have:

   $$\int \sin^7(x)\cos^4(x)\,[\cos(x)\,dx] = \int \sin^7(x)(1-\sin^2(x))^2\,[\cos(x)\,dx]$$

   Make the substitution with $u = \sin(x)$ and we get (expand the square):

   $$\int u^7(1-2u^2+u^4)\,du = \int u^7 - 2u^9 + u^{11}\,du$$

   which is simple to integrate:

   $$\frac{1}{8}\sin^8(x) - \frac{1}{5}\sin^{10}(x) + \frac{1}{12}\sin^{12}(x) + C$$

3. $\int \cos^2(x)\,dx$

   SOLUTION: For an even power use the half angle formula:

   $$\int \frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

4. $\int \sin^4(x)\,dx$

   SOLUTION: Use the half angle formula (twice):

   $$\int \left(\frac{1-\cos(2x)}{2}\right)^2 dx = \int \frac{1}{4}(1 - 2\cos(2x) + \cos^2(2x))\,dx$$

   Use the half angle formula again for $\cos^2(2x) = \dfrac{1+\cos(4x)}{2}$, and we have:

   $$\int \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\,dx = \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

5. $\int x \sin^2(x)\,dx$

There are two issues here: Because of the leading x, we'll need to use "integration by parts", and because of the $\sin^2(x)$, we'll need to use the half angle formula. For the standard integration by parts, we need to assign values to u and dv, then find du and v:

$$u = x \qquad du = 1\,dx$$

$$dv = \sin^2(x)\,dx \qquad \Rightarrow v = \int \sin^2(x)\,dx = \frac{1}{2}\int 1 - \cos(2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x)$$

6. $\int x \sin^3(x)\,dx$