Exam 2 Sample Questions

Be sure to look over your old quizzes and homework as well. For limits, we will provide a graph and contours. No calculators will be allowed for this exam.

1. True or False, and explain:

   (a) There exists a function f with continuous second partial derivatives such that
   $$f_x(x,y) = x + y^2 \qquad f_y = x - y^2$$

   (b) The function f below is continuous at the origin.
   $$f(x,y) = \begin{cases} \frac{2xy}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   (c) If $\vec{r}(t)$ is a differentiable vector function, then
   $$\frac{d}{dt}|\vec{r}(t)| = |\vec{r}\,'(t)|$$

   (d) If $z = 1 - x^2 - y^2$, then the linearization of z at $(1,1)$ is
   $$L(x,y) = -2x(x-1) - 2y(y-1)$$

   (e) We can always use the formula: $\nabla f(a,b)\cdot\vec{u}$ to compute the directional derivative at (a,b) in the direction of $\vec{u}$.

   (f) Different parameterizations of the same curve result in identical tangent vectors at a given point on the curve.

   (g) If $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector functions, then
   $$\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}\,'(t) \times \vec{v}\,'(t)$$

   (h) If $f_x(a,b)$ and $f_y(a,b)$ both exist, then f is differentiable at (a,b).

   (i) At a given point on a curve $(x(t_0), y(t_0), z_0(t))$, the osculating plane through that point is the plane through $(x(t_0), y(t_0), z(t_0))$ with normal vector is $\vec{B}(t_0)$.

2. Show that, if $|\vec{r}(t)|$ is a constant, then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$. (HINT: Differentiate $|\vec{r}(t)|^2 = k$)

3. Reparameterize the curve with respect to arc length measuring from $t = 0$ in the direction of increasing t:
   $$\mathbf{r} = 2t\mathbf{i} + (1-3t)\mathbf{j} + (5+4t)\mathbf{k}$$

4. Is it possible for the directional derivative to exist for every unit vector $\vec{u}$ at some point (a,b), but f is still not differentiable there?

   Consider the function $f(x,y) = \sqrt[3]{x^2y}$. Show that the directional derivative exists at the origin (by letting $\vec{u} = \langle\cos(\theta), \sin(\theta)\rangle$ and using the **definition**), BUT, f is not differentiable at the origin (because if it were, we could use $\nabla f \cdot \vec{u}$ to compute $D_{\vec{u}}f$).

5. If $f(x,y) = \sin(2x+3y)$, then find the linearization of f at $(-3,2)$.

6. The radius of a right circular cone is increasing at a rate of 3.5 inches per second while its height is decreasing at a rate of 4.3 inches per second. At what rate is the volume changing when the radius is 160 inches and the height is 200 inches? ($V = \frac{1}{3}\pi r^2 h$)

7. Find the (total) differential of the function: $v = y\cos(xy)$

8. Find the maximum rate of change of $f(x,y) = x^2y + \sqrt{y}$ at the point $(2,1)$, and the direction in which it occurs.

9. Find an expression for
$$\frac{d}{dt}\left[\mathbf{u}(t)\cdot(\mathbf{v}(t)\times\mathbf{w}(t))\right]$$

10. Use the definition of the partial derivative to compute $f_x(x,y)$, if $f(x,y) = \frac{x}{x+y^2}$.

11. The curves below intersect at the origin. Find the angle of intersection to the nearest degree:
$$\vec{r}_1(t) = \langle t, t^2, t^9\rangle \quad \vec{r}_2(t) = \langle \sin(t), \sin(5t), t\rangle$$

12. Find three positive numbers whose sum is 100 and whose product is a maximum.

13. Find the equation of the tangent plane and normal line to the given surface at the specified point:
$$x^2 + 2y^2 - 3z^2 = 3 \qquad (2,-1,1)$$

14. If $z = x^2 - y^2$, $x = w + 4t$, $y = w^2 - 5t + 4$, $w = r^2 - 5u$, $t = 3r + 5u$, find $\partial z/\partial r$.

15. If $x^2 + y^2 + z^2 = 3xyz$ and we treat z as an implicit function of z, then find $\partial z/\partial x$ and $\partial z/\partial y$.

16. If $\mathbf{a}(t) = -10\mathbf{k}$ and $\mathbf{v}(0) = \mathbf{i}+\mathbf{j}-\mathbf{k}$, $\mathbf{r}(0) = 2\mathbf{i}+3\mathbf{j}$, find the velocity and position vector functions.

17. Find the equation of the normal line through the level curve $4 = \sqrt{5x-4y}$ at $(4,1)$ using a gradient.

18. Find all points at which the direction of fastest change in the function $f(x,y) = x^2+y^2-2x-4y$ is $\vec{i}+\vec{j}$.

19. Find the vectors $\mathbf{T}$ and $\mathbf{N}$ if $\mathbf{r}(t) = \langle \cos(t), \sin(t), ln(\cos(t))\rangle$ at the point $(1,0,0)$.

20. Find and classify the critical points for $f(x,y) = 4 + x^3 + y^3 - 3xy$.

21. Let $z = x^2 + y^2$.
    (a) Draw the level curves for $z = 2, 4, 6$.
    (b) Calculate the gradient at $(2,1)$.

(c) Plot the gradient vector you computed in the previous problem, along with the earlier level curves.

(d) Find the equation of the tangent line to the curve at $(2, 1)$.

(e) Find the equation of the normal line to the curve at $(2, 1)$.

22. Consider the surface where z is implicitly defined by:

$$x^3 + y^3 + z^3 = 9 - 6xyz$$

(a) Find z_x and z_y if $x = 1$ and $y = 1$.

(b) Find the equation of the tangent plane at $(1, 1, 1)$.

23. Find parametric equations of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and $z = 4$. (Hint: In which direction should the tangent line go?)

24. Find and classify the critical points:

$$f(x, y) = x^3 - 3x + y^4 - 2y^2$$

25. Find the curve of intersection between the plane $y + z = 3$ and $x^2 + y^2 = 5$ (in parametric form).

26. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at $(-1, 1, 2)$.

27. The following table gives numerical values of $z = g(x, y)$. Use these to estimate $g_x(2, 5)$ and $g_y(2, 5)$ by taking an average. Also estimate the directional derivative of g at $(2, 5)$ in the direction of $\langle 1, 1 \rangle$

Side Remark: I would give you numbers that would be easier to work with on the exam, since you wouldn't have a calculator. Go ahead and use a calculator for this problem.

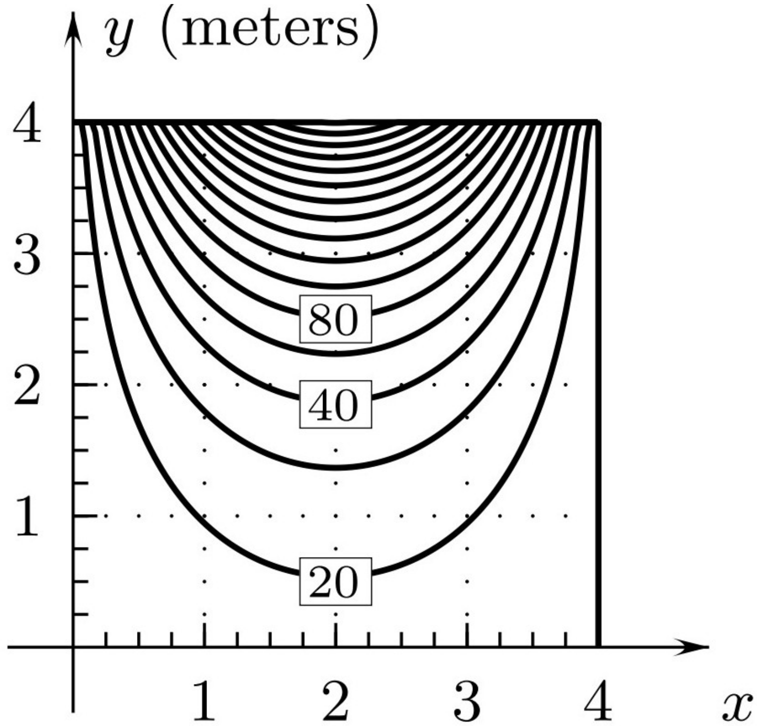
	$x = 1.5$	$x = 2$	$x = 2.5$
$y = 5.2$	16.0	17.2	18.4
$y = 5.0$	20.0	21.2	22.3
$y = 4.8$	24.2	25.3	26.6

28. If $P = \sqrt{u^2 + v^2 + w^2}$, where $u = xe^y$, $v = ye^x$ and $w = e^{xy}$, then find P_x, P_y when $x = 0, y = 2$.

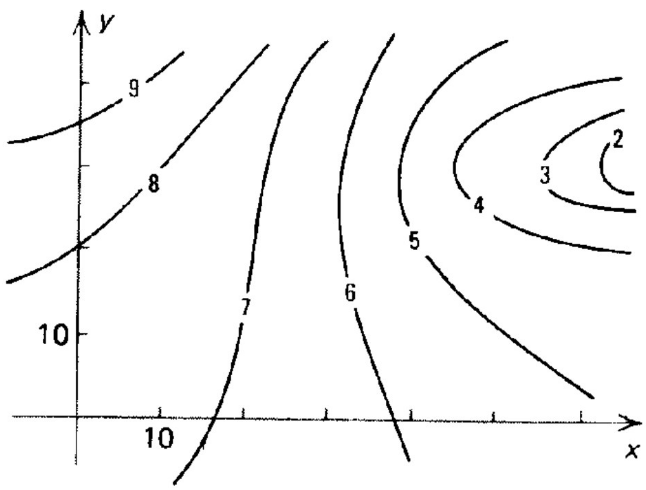
29. Find the rate of change of $f(x, y) = \sqrt{xy}$ at $P(2, 8)$ in the direction of $Q(5, 4)$.

30. Find the points on the surface of $y^2 = 9 + xz$ that are closest to the origin.

31. The figure below shows level curves for the temperature $z = T(x, y)$ on a square plate. First, estimate the values of both partial derivatives of T at $(3, 2)$, then for all of the second partial derivatives, just estimate whether they are positive or negative.



32. The figure below shows the level curves of $z = h(x, y)$. Find whether each is positive or negative: (i) $h_x(50, 30)$ (ii) $h_y(50, 30)$ (iii) $h_{xx}(50, 30)$



33. Using the previous graph, if we make a path from the center of the number "8" always going in the direction of $-\nabla h$, draw the result.