Exam 2 Review Solutions

1. True or False, and explain:

   (a) There exists a function f with continuous second partial derivatives such that

   $$f_x(x,y) = x + y^2 \qquad f_y = x - y^2$$

   SOLUTION: False. If the function has continuous second partial derivatives, then Clairaut's Theorem would apply (and $f_{xy} = f_{yx}$). However, in this case:

   $$f_{xy} = 2y \qquad f_{yx} = 1$$

   (b) The function f below is continuous at the origin.

   $$f(x,y) = \begin{cases} \frac{2xy}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   SOLUTION: Check the limit- First, how about $y = x$ versus $y = -x$?

   $$\lim_{(x,x)\to(0,0)} \frac{2x^2}{3x^2} = \frac{2}{3} \qquad \lim_{(x,-x)\to(0,0)} \frac{-2x^2}{3x^2} = \frac{-2}{3}$$

   Yep, that did it- The limit does not exist at the origin, therefore the function is not continuous at the origin (it is continuous at all other points in the domain).

   (c) If $\vec{r}(t)$ is a differentiable vector function, then

   $$\frac{d}{dt}|\vec{r}(t)| = |\vec{r}\,'(t)|$$

   SOLUTION: False.

   $$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{2}(\mathbf{r}(t)\cdot\mathbf{r}(t))^{-1/2}\left(\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t)\right) = \frac{\mathbf{r}'(t)\cdot\mathbf{r}(t)}{|\mathbf{r}(t)|}$$

   (d) If $z = 1 - x^2 - y^2$, then the linearization of z at $(1,1)$ is

   $$L(x,y) = -2x(x-1) - 2y(y-1)$$

   SOLUTION: False for two reasons. We have forgotten to evaluate the partial derivatives of f at the base point $(1,1)$ (and so the resulting formula is not linear). We have also forgotten to evaluate the function itself at $(1,1)$. The linearization should be:

   $$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = -1 - 2(x-1) - 2(y-1)$$

   (e) We can always use the formula: $\nabla f(a,b)\cdot\vec{u}$ to compute the directional derivative at (a,b) in the direction of $\vec{u}$.
   SOLUTION: False. This formula only works if f is differentiable at (a,b) (See Exercise 4 below).

   (f) Different parameterizations of the same curve result in identical tangent vectors at a given point on the curve.
   SOLUTION: False. The magnitude of $\mathbf{r}(t)$ is the speed. For example, $\mathbf{r}(3t)$ will have a magnitude that is three times that of the original- If you want an actual example, consider

   $$\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$$

   At the point on the unit circle $(1/\sqrt{2}, 1/\sqrt{2})$, the magnitude of $\mathbf{r}'(\pi/4) = 1$. Replace t by $3t$ (and evaluate at $t = \pi/12$ to have the same point on the curve), and the speed is 3 instead of 1.

   Why did we bring this up? If we re-parameterize with respect to *arc length*, the velocity is always 1 unit (so at $s = 1$, you've traveled one unit of length, etc).

(g) If $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector functions, then

$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}\,'(t)\times\vec{v}\,'(t)$$

SOLUTION: False. It looks like the product rule:

$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}\,'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}\,'(t)$$

(h) If $f_x(a,b)$ and $f_y(a,b)$ both exist, then f is differentiable at (a,b).
SOLUTION: False. Our theorem says that in order to conclude that f is differentiable at (a,b), the partial derivatives must be *continuous* at (a,b). Just having the partial derivatives exist at a point is a weak condition- It is not enough to even have continuity.

(i) At a given point on a curve $(x(t_0), y(t_0), z_0(t))$, the osculating plane through that point is the plane through $(x(t_0), y(t_0), z(t_0))$ with normal vector is $\vec{B}(t_0)$.
SOLUTION: True (by definition).

2. Show that, if $|\vec{r}(t)|$ is a constant, then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$. (HINT: Differentiate $|\vec{r}(t)|^2 = k$)
SOLUTION: Using the hint and the fact that $|\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$, we then differentiate both sides:

$$|\vec{r}(t)|^2 = k \quad\Rightarrow\quad \vec{r}(t)\cdot\vec{r}(t) = k \quad\Rightarrow\quad \vec{r}\,'(t)\cdot\vec{r}(t)+\vec{r}(t)\cdot\vec{r}\,'(t) = 0 \quad\Rightarrow\quad 2\vec{r}(t)\cdot\vec{r}\,'(t) = 0$$

Therefore, the dot product is zero (and so $\vec{r}\,'(t)$ and $\vec{r}(t)$ are orthogonal).

3. Reparameterize the curve with respect to arc length measuring from $t=0$ in the direction of increasing t:

$$\mathbf{r} = 2t\mathbf{i} + (1-3t)\mathbf{j} + (5+4t)\mathbf{k}$$

SOLUTION: Find s as a function of t, invert it then substitute it back into the expression so that $\mathbf{r}$ is a function of s. In this case,

$$s = \int_0^t |\mathbf{r}'(u)|\,du = \sqrt{29}\,t$$

Therefore, $t = s/\sqrt{29}$, and

$$\mathbf{r}(s) = \left\langle \frac{2}{\sqrt{29}}s, 1-\frac{3}{\sqrt{29}}s, 5+\frac{4}{\sqrt{29}}s\right\rangle$$

4. Is it possible for the directional derivative to exist for every unit vector $\vec{u}$ at some point (a,b), but f is still not differentiable there?

Consider the function $f(x,y) = \sqrt[3]{x^2y}$. Show that the directional derivative exists at the origin (by letting $\vec{u} = \langle\cos(\theta),\sin(\theta)\rangle$ and using the **definition**), BUT, f is not differentiable at the origin (because if it were, we could use $\nabla f\cdot\vec{u}$ to compute $D_{\vec{u}}f$).

SOLUTION: Compute the directional derivative at the origin by using the definition:

$$D_uf(0,0) = \lim_{h\to 0}\frac{f(0+h\cos(\theta), 0+h\sin(\theta)) - f(0,0)}{h} = \lim_{h\to 0}\frac{h\sqrt[3]{\cos^2(\theta)\sin(\theta)}}{h} = \sqrt[3]{\cos^2(\theta)\sin(\theta)}$$

By using the definition, the directional derivative exists for all angles θ. In particular, $\theta = 0$ corresponds to f_x, and $\theta = \pi/2$ corresponds to f_y, so that the partial derivatives also exist at the origin: $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

If we use our shortcut formula to compute $D_uf(0,0)$, we get zero for every direction:

$$D_uf = \nabla f\cdot\vec{u} = 0$$

which is not true.

Why did this happen? If we compute the first partial derivatives,

$$f_x = \frac{2}{3}x^{-1/3}y^{1/3} \qquad f_y = \frac{1}{3}x^{2/3}y^{-2/3}$$

we see that f_x and f_y are both discontinuous where $x = 0$ and $y = 0$ respectively.

5. If $f(x, y) = \sin(2x + 3y)$, then find the linearization of f at $(-3, 2)$.

   SOLUTION: We have $f(-3, 2) = \sin(0) = 0$ and

$$f_x(x, y) = 2\cos(2x + 3y) \quad \Rightarrow \quad f_x(-3, 2) = 2$$
$$f_y(x, y) = 3\cos(2x + 3y) \quad \Rightarrow \quad f_y(-3, 2) = 3$$

   Therefore,

$$L(x, y) = 0 + 2(x + 3) + 3(y - 2) = 2(x + 3) + 3(y - 2)$$

6. The radius of a right circular cone is increasing at a rate of 3.5 inches per second while its height is decreasing at a rate of 4.3 inches per second. At what rate is the volume changing when the radius is 160 inches and the height is 200 inches? ($V = \frac{1}{3}\pi r^2 h$)

   SOLUTION:

$$\frac{dV}{dt} = V_r \frac{dr}{dt} + V_h \frac{dh}{dt} \quad \Rightarrow \quad \frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \, \frac{dh}{dt}$$

   Use $r = 160$, $h = 200$, $dr/dt = 3.5$ and $dh/dt = -4.3$, $dV/dt \approx 37973.3\pi$. I'll try to use nicer numbers for the exam.

7. Find the differential of the function: $v = y\cos(xy)$

   SOLUTION:

$$dv = v_x \, dx + v_y \, dy = (-y^2 \sin(xy) \, dx + (\cos(xy) - xy \sin(xy)) \, dy$$

8. Find the maximum rate of change of $f(x, y) = x^2 y + \sqrt{y}$ at the point $(2, 1)$, and the direction in which it occurs.

   SOLUTION: The maximum rate of change occurs if we move in the direction of the gradient. We see this by recalling that:

$$D_u f = \nabla f \cdot \vec{u} = |\nabla f| \cos(\theta)$$

   so we find the gradient at the point $(2, 1)$

$$\nabla f = \langle 4, 9/2 \rangle$$

   So if we move in that direction, the we get the max rate of change, which is

$$|\nabla f| = \sqrt{4^2 + \frac{81}{4}} = \frac{\sqrt{145}}{2} \approx 6.02$$

9. Find an expression for

$$\frac{d}{dt}\left[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t)\right]$$

   SOLUTION:

$$\frac{d}{dt}\left[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t)\right] = \mathbf{u}' \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})'$$

   Taking this derivative, we see that

$$\frac{d}{dt}\left[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t)\right] = \mathbf{u}' \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot (\mathbf{v}' \times \mathbf{w} + \mathbf{v} \times \mathbf{w}')$$

10. Use the definition of the partial derivative to compute $f_x(x,y)$, if $f(x,y) = \frac{x}{x+y^2}$.

SOLUTION:

$$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0} \frac{\frac{x+h}{x+h+y^2} - \frac{x}{x+y^2}}{h}$$

If we work with the numerator a bit, we combine fractions by getting a common denominator:

$$\frac{x+h}{x+h+y^2} - \frac{x}{x+y^2} = \frac{(x+h)(x+y^2) - x(x+h+y^2)}{(x+h+y^2)(x+y^2)} = \frac{hy^2}{(x+h+y^2)(x+y^2)}$$

Put this back in,

$$\lim_{h\to 0} \frac{1}{h} \frac{hy^2}{(x+h+y^2)(x+y^2)} = \frac{y^2}{(x+y^2)^2}$$

11. The curves below intersect at the origin. Find the angle of intersection to the nearest degree:

$$\vec{r}_1(t) = \langle t, t^2, t^9 \rangle \quad \vec{r}_2(t) = \langle \sin(t), \sin(5t), t \rangle$$

SOLUTION:

The angle of intersection is the angle between the tangent vectors at the origin. First differentiate, then evaluate at $t = 0$:

$$\vec{r}_1{}'(t) = \langle 1, 2t, 9t^8 \rangle \quad \vec{r}_2{}'(t) = \langle \cos(t), 5\cos(5t), 1 \rangle \quad \Rightarrow \quad \langle 1,0,0 \rangle, \langle 1,5,1 \rangle$$

To find the angle, we use the relationship:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos(\theta)$$

In our case:

$$\cos(\theta) = \frac{1}{\sqrt{1^2 + 5^2 + 1^2}} \quad \Rightarrow \quad \theta = \cos^{-1}(1/\sqrt{27}) \approx 79^\circ$$

12. Find three positive numbers whose sum is 100 and whose product is a maximum.

SOLUTION: Let x, y, z be the three numbers. Then we want to find the maximum of $P(x,y,z) = xyz$ subject to the constraint that $x + y + z = 100$ and they are all positive.

Let $P(x,y) = xy(100 - x - y)$. The critical points are

$$\begin{aligned} y(100 - x - y) - xy &= 0 \\ x(100 - x - y) - xy &= 0 \end{aligned}$$

From the first equation, $y = 100 - 2x$ (we can throw out $y = 0$). Substitute this into the second equation to find that $x = 100/3$. Therefore, $y = 100/3$ and $z = 100/3$. These are the three numbers we wanted.

13. Find the equation of the tangent plane and normal line to the given surface at the specified point:

$$x^2 + 2y^2 - 3z^2 = 3 \qquad (2, -1, 1)$$

SOLUTION: This is an implicitly defined surface of the form $F(x,y,z) = k$, therefore, we know that ∇F is orthogonal to the tangent planes on the surface. Compute ∇F at $(2,-1,1)$, and construct the plane and line:

$$F_x = 2x \quad F_y = 4y \quad F_z = -6z \quad \Rightarrow \quad \nabla F(2,-1,1) = \langle 4, -4, -6 \rangle$$

Thus, the tangent plane is:

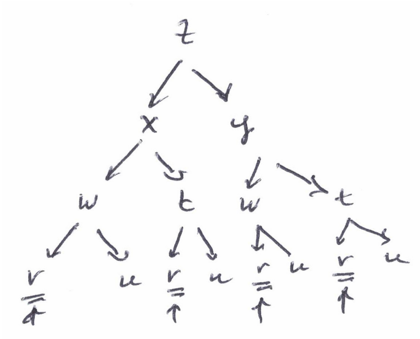
$$4(x-2) - 4(y+1) - 6(z-1) = 0$$

The normal line goes in the direction of the gradient, starting at the given point. In parametric form,

$$x(t) = 2 + 4t \quad y(t) = -1 - 4t \quad z(t) = 1 - 6t$$

14. If $z = x^2 - y^2$, $x = w + 4t$, $y = w^2 - 5t + 4$, $w = r^2 - 5u$, $t = 3r + 5u$, find $\partial z/\partial r$.

SOLUTION: This is simple if we use a chart (see below):



$$z_r = z_x x_w w_r + z_x x_t t_r + z_y y_w w_r + z_y y_t t_r$$

With:

$$z_x = 2x \quad z_y = -2y$$

$$x_w = 1 \quad x_t = 4 \qquad y_w = 2w \quad y_t = -5$$

$$w_r = 2r \quad t_r = 3$$

so that:

$$z_r = 4xr + 24x - 8ywr + 30y$$

15. If $x^2 + y^2 + z^2 = 3xyz$ and we treat z as an implicit function of x, y, then find $\partial z/\partial x$ and $\partial z/\partial y$.

SOLUTION: Let us define $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz$ in keeping with the notation from the text. Then we compute:

$$F(x,y,z) = 0 \Rightarrow F_x \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(2x - 3yz)}{2z - 3xy}$$

Similarly, we can show that

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(2y - 3xz)}{2z - 3xy}$$

16. If $\mathbf{a}(t) = -10\mathbf{k}$ and $\mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j}$, find the velocity and position vector functions.

SOLUTION:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \langle 0, 0, -10t \rangle + \mathbf{v}_0 = \langle 1, 1, -10t - 1 \rangle$$

And antidifferentiate once more:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \langle t, t, -5t^2 - t \rangle + \mathbf{r}_0 = \langle t + 2, t + 3, -5t^2 - t \rangle$$

17. Find the equation of the normal line through the level curve $4 = \sqrt{5x - 4y}$ at $(4, 1)$ using a gradient.

SOLUTION: The gradient of g is orthogonal to its level curve at $\sqrt{5x - 4y} = 4$. Find the gradient of g at $(4, 1)$:

$$\nabla g = \frac{1}{8}\langle 5, -4 \rangle$$

For the line, we simply need to move in the direction of the gradient, so we can simplify the direction to $\langle 5, -4 \rangle$ (not necessary, but easier for the algebra).

Therefore, the line (in parametric and symmetric form) is:

$$x(t) = 4 + 5t \quad y(t) = 1 - 4t \qquad \text{or} \qquad \frac{x - 4}{5} = \frac{y - 1}{-4}$$

Notice that the slope is $-4/5$. If we wanted to check our answer, we could find the slope of the tangent line:

$$5x - 4y = 16 \quad \Rightarrow \quad 5 - 4\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{5}{4}$$

18. Find all points at which the direction of fastest change in the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\vec{i} + \vec{j}$.

SOLUTION: The direction of the fastest increase is in the direction of the gradient. Therefore, another way to phrase this question is: When is the gradient pointing in the direction of $\langle 1, 1\rangle$:

$$\nabla f = k\langle 1, 1\rangle \quad \Rightarrow \quad \langle 2x - 2, 2y - 4\rangle = \langle k, k\rangle$$

So $k = 2x - 2$ and $k = 2y - 4$, therefore, the points are on the line $2x - 2 = 2y - 4$, or $y = x + 1$. Our conclusion: There are an infinite number of possibilities- All of the form $(a, a + 1)$, which result in the gradient:

$$(2a - 2)\langle 1, 1, \rangle \qquad a > 1$$

19. Find the vectors $\mathbf{T}$ and $\mathbf{N}$ if $\mathbf{r}(t) = \langle \cos(t), \sin(t), ln(\cos(t))\rangle$ at the point $(1, 0, 0)$.

SOLUTION: Recall the definitions:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

With the given function $\mathbf{r}$, we have:

$$\mathbf{r}' = \left\langle -\sin(t), \cos(t), -\frac{\sin(t)}{\cos(t)}\right\rangle$$

And the magnitude is (we're assuming $\cos(t) > 0$ for the log to be defined):

$$|\mathbf{r}'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + \tan^2(t)} = \sqrt{1 + \tan^2(t)} = \sqrt{\sec^2(t)} = |\sec(t)| = \sec(t) = \frac{1}{\cos(t)}$$

Now

$$\mathbf{T}(t) = \langle -\sin(t)\cos(t), \cos^2(t), -\sin(t)\rangle$$

For $\mathbf{N}$, we differentiate $\mathbf{T}$ and normalize it:

$$\mathbf{T}' = \left\langle -\cos^2(t) + \sin^2(t), -2\sin(t)\cos(t), -\cos(t)\right\rangle$$

with

$$|\mathbf{T}'|^2 = (-\cos^2(t) + \sin^2(t))^2 + 4\sin^2(t)\cos^2(t) + \cos^2(t)$$

If you square this out and combine the first two terms, we get:

$$\cos^4(t) - 2\cos^2(t)\sin^2(t) + \sin^4(t) + 4\sin^2(t)\cos^2(t) =$$

$$\cos^4(t) + 2\cos^2(t)\sin^2(t) + \sin^4(t) = (\cos^2(t) + \sin^2(t))^2 = 1$$

so we end up with:

$$|\mathbf{T}'|^2 = 1 + \cos^2(t)$$

(You wouldn't have to go this far to simplify on an exam, but its good practice). Now we can write

$$\mathbf{N}(t) = \frac{1}{\sqrt{1 + \cos^2(t)}}\left\langle -\cos^2(t) + \sin^2(t), -2\sin(t)\cos(t), -\cos(t)\right\rangle$$

20. Find and classify the critical points:

$$f(x, y) = 4 + x^3 + y^3 - 3xy$$

SOLUTION: The partial derivatives are:

$$f_x = 3x^2 - 3y \quad f_y = 3y^2 - 3x \quad f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = -3$$

The critical points are where $x^2 = y$ and $y^2 = x$, so x, y are both zero or positive:

$$x^4 = x \quad \Rightarrow \quad x^4 - x = 0 \quad \Rightarrow \quad x(x^3 - 1) = 0$$

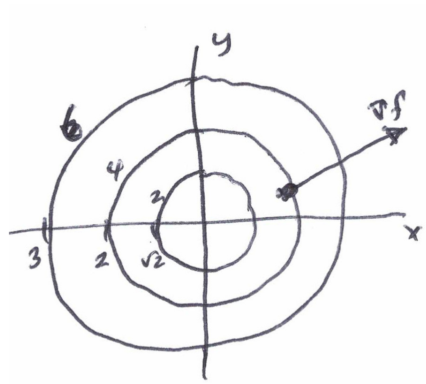
so $x = 0, y = 0$ or $x = 1, y = 1$. Put these points into the Second Derivatives Test:

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = -9 < 0 \quad \Rightarrow \quad \text{The origin is a SADDLE}$$

$$f_{xx}(1,1)f_{yy}(1,1) - f_{xy}^2(1,1) = 36 - 9 > 0 \quad f_{xx}(1,1) > 0 \Rightarrow \quad \text{Local MIN}$$

21. Let $z = x^2 + y^2$.
   (a) Draw the level curves for $z = 2, 4, 6$.

   SOLUTION: See the sketch. The level curves are circles.



   (b) Calculate the gradient at $(2, 1)$.

   SOLUTION: $\nabla f(2, 1) = \langle 4, 2\rangle$.

   (c) Plot the gradient vector you computed in the previous problem, along with the earlier level curves.

   SOLUTION: I'm mainly looking to see that you drew it perpendicular to the level curves, and the gradient vector is fairly long at that point.

   (d) Find the equation of the tangent line to the curve at $(2, 1)$.
   SOLUTION: The slope is $-f_x/f_y$, or from the gradient, -2.

$$y - 1 = -2(x - 2)$$

   (e) Find the equation of the normal line to the curve at $(2, 1)$.
   SOLUTION: The slope is the negative reciprocal, or $1/2$.

$$y - 1 = \frac{1}{2}(x - 2)$$

   You could write this in parametric vector form using the gradient vector:

$$\langle 2 + 4t, 1 + 2t\rangle$$

22. Find the equation of the tangent plane to the surface implicitly defined below at the point $(1, 1, 1)$:

$$x^3 + y^3 + z^3 = 9 - 6xyz$$

SOLUTION: First write this as $F(x, y, z) = 0$

$$F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 9 = 0$$

Now, for z_x at $(1, 1, 1)$:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6y\left(z + x\frac{\partial z}{\partial x}\right) = 0$$

At the point $(1,1,1)$ we have:

$$3 + 3z_x + 6(1 + z_x) = 0 \quad \Rightarrow \quad z_x = \frac{-9}{9} = -1$$

Similarly, for z_y:

$$3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6x\left(z + y\frac{\partial z}{\partial y}\right) = 0$$

so that $z_y = -1$ as well. For the tangent plane, compute the gradient for the normal vector:

$$F_x = 3x^2 + 6yz \quad F_y = 3y^2 + 6xz \quad F_z = 3z^2 + 6xy \quad \Rightarrow \quad \nabla F = \langle 9, 9, 9\rangle$$

Therefore, the tangent plane is:

$$9(x-1) + 9(y-1) + 9(z-1) = 0$$

(You could divide by 9 if you like).

23. Find parametric equations of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and $z = 4$. (Hint: In which direction should the tangent line go?)

The curve is $2x^2 - y^2 = 4$, which is an ellipse (at height 4 in 3-d). The gradient is $\langle 4x, -2y\rangle$ so at $(-2, 2)$, the gradient is $< -8, -4 >$, so the tangent line (in the $xy-$ plane) is:

$$-8(x+2) - 4(y-2) = 0 \quad \text{or} \quad y = -2(x+2) + 2$$

We should express the line in three dimensions, since the question was phrased that way. Here's one way:

$$\langle t, -2(t+2) + 2, 4\rangle$$

(But there are multiple ways of expressing it).

24. Find and classify the critical points:

$$f(x, y) = x^3 - 3x + y^4 - 2y^2$$

SOLUTION: We use the second derivatives test to classify the critical points as local min, local max or saddle.

Solving for the CPs, we get:

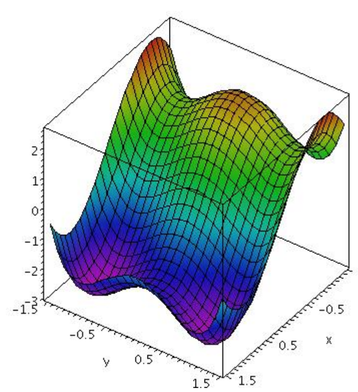
$$f_x(x, y) = 3x^2 - 3 = 0 \qquad f_y(x, y) = 4y^3 - 4y = 0$$

from which we get $x = \pm 1, y = \pm 1$ and $x = \pm 1, y = 0$ Continuing with second derivatives,

$$D(x, y) = \begin{vmatrix} f_x x & f_x y \\ f_y x & f_y y \end{vmatrix} = \begin{vmatrix} 6x & 0 \\ 0 & 12y^2 - 4 \end{vmatrix} = 24x(3y^2 - 1)$$

We'll arrange the results in a list. The plot is included just for fun:

Point	D and Classification
$(1,1)$	$48:$ Local Min
$(1,-1)$	$48:$ Local Min
$(1,0)$	$-24:$ Saddle
$(-1,1)$	$-48:$ Saddle
$(-1,-1)$	$-48:$ Saddle
$(-1,0)$	$24:$ Local Max



25. Find the curve of intersection between the plane $y + z = 3$ and $x^2 + y^2 = 5$ (in parametric form).

SOLUTION 1: Since we have a circle, let

$$x(t) = \sqrt{5}\cos(t), \quad y(t) = \sqrt{5}\sin(t), \quad z(t) = 3 - \sqrt{5}\sin(t)$$

with $0 \le t \le 2\pi$.

SOLUTION 2: An alternative that's not quite as nice would be to write x and z as functions of y. It's not as nice because x is not a function of t, so you need to write two curves:

$$\begin{array}{ll} x = \sqrt{5 - y^2} & \qquad x = -\sqrt{5 - y^2} \\ y = y & \qquad y = y \\ z = 3 - y & \qquad z = 3 - y \end{array}$$

26. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at $(-1, 1, 2)$.

SOLUTION: We could go through and find the curve of intersection, but notice that we are not asked to do that. We are given the point of intersection, so we just need a tangent vector.

We make the observation that the tangent line will lie in the tangent plane to both surfaces, so that the direction we need is orthogonal to the normal vector for both tangent planes.

- The normal vector for the tangent plane to $x^2 + y^2 - z = 0$ is $\langle 2x, 2y, -1\rangle$ at $(-1, 1, 2)$, so

$$\mathbf{n}_1 = \langle -2, 2, -1\rangle$$

- Similarly, the normal vector for the tangent plane to $4x^2 + y^2 + z^2 = 9$ will be the gradient evaluated at $(-1, 1, 2)$:

$$\mathbf{n}_2 = \langle -8, 2, 4\rangle \quad \Rightarrow \quad \mathbf{n}_2 = \langle -4, 1, 2\rangle$$

(You didn't need to scale this, but it may simplify your work later)

Taking the cross product, we get the direction for the tangent line, so that the equation in parametric vector form is:

$$\langle -1, 1, 2\rangle + t\langle 5, 8, 6\rangle$$

27. The following table gives numerical values of $z = g(x, y)$. Use these to estimate $g_x(2, 5)$ and $g_y(2, 5)$ by taking an average. Also estimate the directional derivative of g at $(2, 5)$ in the direction of $\langle 1, 1\rangle$

Side Remark: I would give you numbers that would be easier to work with on the exam, since you wouldn't have a calculator. Go ahead and use a calculator for this problem.

	$x = 1.5$	$x = 2$	$x = 2.5$
$y = 5.2$	16.0	17.2	18.4
$y = 5.0$	20.0	21.2	22.3
$y = 4.8$	24.2	25.3	26.6

SOLUTION: TYPO for the directional derivative. We can't really go in the direction of $\langle 1, 1\rangle$ using the table, so that really should have been $\langle 0.5, 0.2\rangle$ or something scaling like that. So we'll take that as the direction below.

Continuing, for $g_x(2, 5)$, take the average change horizontally:

$$\frac{\frac{21.2-20.0}{2-1.5} + \frac{22.3-21.2}{2.5-2.0}}{2} = \frac{2.4 + 2.2}{2} = 2.3$$

Similarly, we estimate g_y the same way, except go vertically:

$$\frac{\frac{21.2-25.3}{5-4.8}+\frac{17.2-21.2}{5.2-5.0}}{2}=\frac{-20.5+(-20)}{2}=-20.25$$

And we estimate the directional derivative going diagonally. The esimation is a bit different, we are taking the change in height over the change in the distance in the domain, which is the distance between the points $(2.5, 5.2)$ and $(2.0, 5.0)$, which is approximately 0.54:

$$\frac{\frac{18.4-21.2}{0.54}+\frac{21.2-24.2}{0.54}}{2}\approx -5.37$$

28. If $P=\sqrt{u^2+v^2+w^2}$, where $u=xe^y$, $v=ye^x$ and $w=e^{xy}$, then find P_x, P_y when $x=0, y=2$.

    SOLUTION: Using a chart can help with the notation:

    $$P_x = P_u u_x + P_v v_x + P_w w_x \qquad P_y = P_u u_y + P_v v_y + P_w w_y$$

    Taking all of the partial derivatives and evaluating, we should get that

    $$P_x=\frac{8}{\sqrt{5}} \qquad P_y=\frac{2}{\sqrt{5}}$$

29. Find the rate of change of $f(x,y)=\sqrt{xy}$ at $P(2,8)$ in the direction of $Q(5,4)$.

    We need the partial derivatives to compute the gradient:

    $$f_x=\left.\frac{\sqrt{y}}{2\sqrt{x}}\right|_{x=2,y=8}=1 \qquad f_y=\left.\frac{\sqrt{x}}{2\sqrt{y}}\right|_{x=2,y=8}=\frac{1}{4}$$

    We note that the partial derivatives are continuous at P, so f is differentiable and our shortcut formula for the directional derivative will work. We now need a unit vector in the right direction. The direction we want is:

    $$\overrightarrow{PQ}=\langle 5-2, 4-8\rangle=\langle 3,-4\rangle \quad \Rightarrow \quad \mathbf{u}=\frac{1}{5}\langle 3,-4\rangle$$

    The answer is now the directional derivative:

    $$D_u f=\left\langle 1,\frac{1}{4}\right\rangle\cdot\left\langle \frac{3}{5},-\frac{4}{5}\right\rangle=\frac{2}{5}$$

30. Find the points on the surface of $y^2=9+xz$ that are closest to the origin.

    SOLUTION: When we optimize the distance, we can instead optimize the square of the distance- The location where the max or min occurs will not change, although you'll need to take the square root of the max or min at the end.

    In this case, we want to minimize: $(x-0)^2+(y-0)^2+(z-0)^2$ where the point (x,y,z) must lie on the surface: $y^2=9+xz$. Substituting this in, we want to find the minimum of:

    $$F(x,z)=x^2+(9+xz)+z^2$$

    We'll find the critical points first:

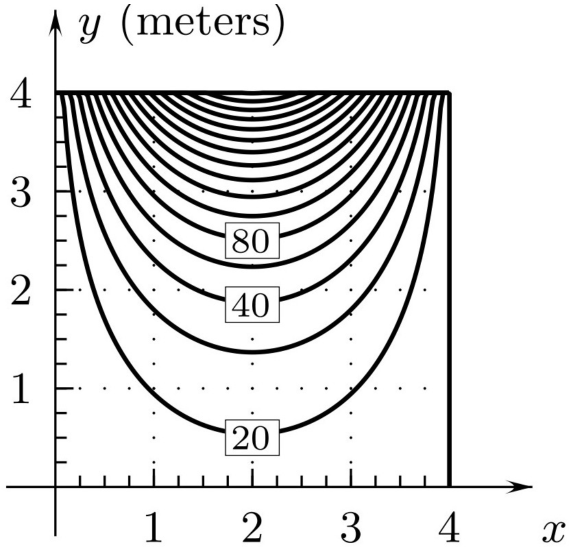
    $$2x+z=0 \qquad x+2z=0$$

    From these, the only critical point is the origin $(0,0)$. Is this a min or a max? We should do a quick check using the second derivatives test-

    $$\begin{array}{ll} F_{xx}=2 & F_{xz}=1 \\ F_{zx}=1 & F_{zz}=2 \end{array} \quad \Rightarrow \quad D=3>0 \text{ and } F_{xx}>0$$

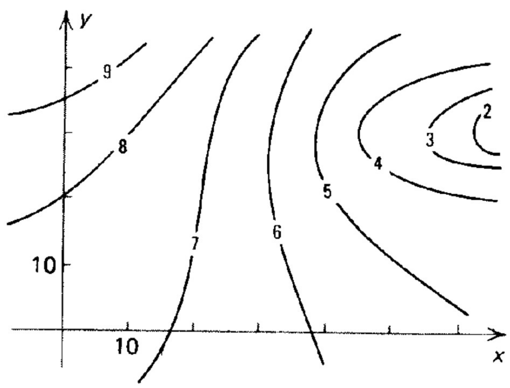
    Therefore, this is a local min, so from that we can conclude that we have the minimum. The points on the surface giving us the minimum are $(0,\pm 3,0)$, which each has a distance of 3.

31. The figure below shows level curves for the temperature $z = T(x, y)$ on a square plate. First, estimate the values of both partial derivatives of T at $(3, 2)$, then for all of the second partial derivatives, just estimate whether they are positive or negative.



- For T_x, a rough estimate would be about -20. Similarly, a rough estimate for T_y would be about $+20$.
- For T_{xx}, T_x goes from about -25 to about -14 as we increase x, so that would mean that $T_{xx} > 0$.
- For T_{yy}, as we go up, T_y increases, so $T_{yy} > 0$.
- For T_{xy}, we might estimate $T_x(3, 2) \approx -20$, and $T_x(3, 2.25) \approx -40$, so $T_{xy} < 0$.

32. The figure below shows the level curves of $z = h(x, y)$. Find whether each is positive or negative: (i) $h_x(50, 30)$ (ii) $h_y(50, 30)$ (iii) $h_{xx}(50, 30)$



- h is decreasing as we increase x, so $h_x < 0$.
- h is decreasing as we increase y (by a small amount), so $h_y < 0$.
- The level curves are getting wider apart as we move in the positive x, so h_x is getting less negative- $h_{xx} > 0$.

33. Using the previous graph, if we make a path from the center of the number "8" always going in the direction of $-\nabla h$, draw the result.

SOLUTION: The main point here is that whenever you cross a level curve, the path is orthogonal to the level curve.