Exam 1 Review SOLUTIONS

1. True or False (and give a short reason):

   (a) If the parametric curve $x = f(t), y = g(t)$ satisfies $g'(1) = 0$, then it has a horizontal tangent line when $t = 1$.

   FALSE: To make the statement true, we would need to know that $f'(1) \neq 0$ (in the denominator of dy/dx).

   (b) The equations $r = 2$, $x^2+y^2 = 4$, and the set of parametric equations: $(x(t), y(t)) = (2\sin(3t), 2\cos(3t))$ for $0 \leq t \leq 2\pi$, all have the same graph.

   TRUE: These would all create a circle of radius 2 centered at the origin in the (x, y) plane.

   (c) For the next set of questions, assume $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are arbitrary vectors in three dimensions (book notation is V_3, in class we said $\mathbb{R}^3$).

      i. $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$
      FALSE in general- You can pick just about any two vectors to show this. For example, $\mathbf{i}, \mathbf{j}$:
      $$\sqrt{2} \neq 1 + 1$$

      ii. $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$
      FALSE: Again, this in false for just about any two vectors. Using $\mathbf{i}, \mathbf{j}$ again:
      $$0 \neq 1 \cdot 1$$

      iii. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
      TRUE. The cross product does distribute over a sum.

      iv. $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$
      FALSE: You could make it true by multiplying the right hand side of the equation by -1.

      v. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$
      FALSE. We showed this in the Maple worksheet.

2. Eliminate the parameter to find a Cartesian equation for the curve: $x = \mathrm{e}^t$, $y = 2\mathrm{e}^t$.

   SOLUTION: $y = 2x$

   Find the arc length of the curve from $t = 1$ to $t = 2$ of the original parametric form, then for the (x, y) form.

   SOLUTION:

$$L = \int_{t=1}^{t=2} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_{t=1}^{t=2} \sqrt{5}\mathrm{e}^t\, dt = \sqrt{5}(\mathrm{e}^2 - \mathrm{e})$$

$$L = \int_{x=\mathrm{e}^1}^{x=\mathrm{e}^2} \sqrt{1 + (f'(x))^2}\, dx = \int_{\mathrm{e}}^{\mathrm{e}^2} \sqrt{1+4}\, dx = \sqrt{5}(\mathrm{e}^2 - \mathrm{e})$$

3. Let $x = t^2 - 2t$ and $y = t + 1$

(a) At what point does the curve cross the x and y axes?

SOLUTION: For the x axis, we check where $y = 0$, and that gives time $t = -1$. The point is then $(3, 0)$.

Similarly for the y axis, we check where $x = 0$, and that occurs when $t = 0$ and $t = 2$. The two points are $(0, 1)$ and $(0, 3)$.

(b) Compute dy/dx and d^2y/dx^2.

SOLUTION:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2t-2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(1/(2t-2))}{2t-2} = -\frac{2}{(2t-2)^3}$$

(c) Where does the curve have a horizontal tangent line?

SOLUTION: We see that dy/dx is never zero, so the curve does not have a horizontal tangent line.

(d) Where does the curve have a vertical tangent line?

SOLUTION: There is a vertical tangent line at $t = 1$.

(e) Eliminate the parameter to get x as a function of y.

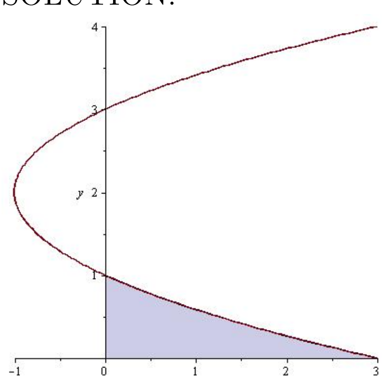
SOLUTION: We see that $t = y - 1$, so substituting that into the expression for x, we have:

$$x = (y-1)^2 - 2(y-1) = (y-1)(y-3) = y^2 - 4y + 3$$

This is a sideways parabola (opening to the right, since y^2 is positive. We found the $x-$ and $y-$intercepts previously, so we could sketch this if we wanted (for the next question).

(f) Find the area in the *first* quadrant bounded by the curve and the coordinate axes (write your integral in terms of t).

SOLUTION:



Note time: At $(0, 1)$, time is $t = 0$ and at $(3, 0)$, $t = -1$:

$$\int x\,dy = \int_{t=-1}^{t=0} t^2 - 2t\,dt = \frac{4}{3}$$

$$\int y\,dx = \int_{t=0}^{t=-1} (t+1)(2t-2)\,dt = -\int_{t=-1}^{t=0} 2t^2 - 2\,dt = \frac{4}{3}$$

4. A plane is given as $2x + 3y + 4z = 12$. Sketch it in the first octant by drawing the lines of intersection between it and the coordinate planes.

   Give the equations of the lines of intersection.

   SOLUTION: There are multiple ways of writing the three lines in parametric form. One way is to form points P, Q, R from the coordinate axes:

$$P(6,0,0) \quad Q(0,4,0) \quad R(0,0,3)$$

   We see that the vector $\overrightarrow{QP} = \langle 6, -4, 0\rangle$ is in the xy-plane, so that (using P), the line is given by:

$$x = 6 + 6t \quad y = -4t \quad z = 0$$

   Using the vector $\overrightarrow{QR} = \langle 0, -4, 3\rangle$ in the yz-plane, using point R:

$$x = 0 \quad y = -4t \quad z = 3 + 3t$$

   Thirdly, the vector $\overrightarrow{RP} = \langle -6, 0, 3\rangle$ in the xz-plane, using point P would be:

$$x = 6 - 6t \quad y = 0 \quad z = 3t$$

5. Find the equation of the plane through the points $(3, -1, 1)$, $(4, 0, 2)$ and $(6, 3, 1)$.

   SOLUTION: There are two ways of expressing the plane- We'll go through both. For notation, let the three points be P, Q and R, respectively.

   - In parametric form, we need a point and two vectors:

$$\vec{P} + t\overrightarrow{PQ} + s\overrightarrow{QR} = \langle 3, -1, 1\rangle + t\langle 1, 1, 1\rangle + s\langle 2, 3, -1\rangle$$

   - Using the point P and the normal, we'll need the cross product:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 3 & -1 \end{vmatrix} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

     Therefore, we can write the plane as:

$$-4(x - 3) + 3(y + 1) + (z - 1) = 0$$

6. Convert the polar equation to Cartesian:

   SOLUTION:

   For $r = \tan(\theta)\sec(\theta)$, we might convert to sines/cosines first.

$$r = \frac{\sin(\theta)}{\cos^2(\theta)} \quad \Rightarrow \quad r\cos^2(\theta) = \sin(\theta)$$

Now replace $\cos(\theta) = x/r$ and $\sin(\theta) = y/r$:

$$r\frac{x^2}{r^2} = \frac{y}{r} \quad \Rightarrow \quad x^2 = y$$

For $r = 2\cos(\theta)$, we know that $x = r\cos(\theta)$, so that $\cos(\theta) = x/r$:

$$r = 2\frac{x}{r} \quad \Rightarrow \quad r^2 = 2x \quad \Rightarrow \quad x^2 + y^2 = 2x$$

It would be fine to stop here, but we can also show that this is the equation of a circle:

$$(x^2 - 2x + 1) + y^2 = 1 \quad \Rightarrow \quad (x-1)^2 + y^2 = 1$$

7. Convert the equation from Cartesian to polar:

   SOLUTIONS:

   For $y = 1 + 3x$, we make the substitutions $x = r\cos(\theta$ and $y = r\sin(\theta)$ so that

$$r\sin(\theta) = 1 + 3r\cos(\theta)$$

   Normally, we ought to solve for r, but that wasn't specified. In that case, we would have:

$$r(\sin(\theta) - 3\cos(\theta)) = 1 \quad \Rightarrow \quad r = \frac{1}{\sin(\theta) - 3\cos(\theta)}$$

   And for $x = 4y^2$, we would have:

$$r\cos(\theta) = 4r^2\sin^2(\theta)$$

   Solving for r, we would have: $r = \dfrac{\cos(\theta)}{4\sin^2(\theta)}$

8. Find the slope of the tangent line to the given polar curve at the point specified by θ:

$$r = 2 - \sin(\theta) \qquad \theta = \frac{\pi}{3}$$

   SOLUTION: We want to first write x, y in terms of θ (or t), then using the formula(s) we had for the derivative in parametric form:

$$x = r\cos(\theta) = (2 - \sin(\theta))\cos(\theta) = 2\cos(\theta) - \sin(\theta)\cos(\theta)$$

   and

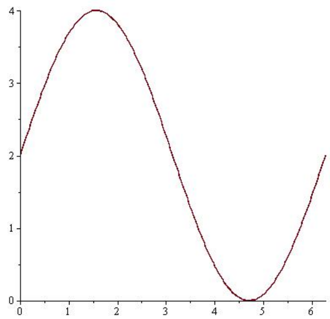
$$y = r\sin(\theta) = (2 - \sin(\theta)\sin(\theta) = 2\sin(\theta) - \sin^2(\theta)$$

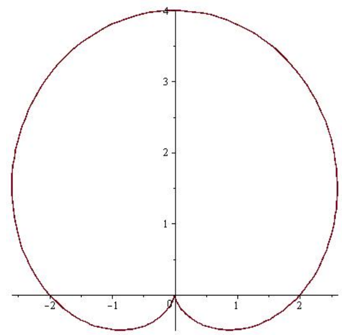
   After some algebra and trig,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 - \sqrt{3}}{1 - 2\sqrt{3}}$$

9. Plot the graph of $r = 2 + 2\sin(\theta)$ in the (r, θ) plane, then transfer that graph to polar coordinates (in the (x, y) coordinate system).

   SOLUTION:





10. Find the distance between the point $(1, 2, 3)$ and the plane $x + y + z = 1$.

    SOLUTION: Distance is:

$$\frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|1 + 2 + 3 - 1|}{\sqrt{3}} = \frac{5}{\sqrt{3}}$$

11. Find the distance between the planes $x + y + z = 1$ and $3x + 3y + 3z - 5 = 0$.

    SOLUTION: Do a quick check that they are parallel. You may use a shortcut formula from the exercises, or take any point from one plane, and find the distance to the second. For example, $(1, 0, 0)$ is on the first plane. The distance to the second is:

$$\frac{|3 - 5|}{\sqrt{3^2 + 3^2 + 3^2}} = \frac{2}{\sqrt{27}} = \frac{2}{3\sqrt{3}}$$

12. Find the angle between the vectors $\langle 1, 2, 3 \rangle$ and $\langle 1, 0, 1 \rangle$. Find the projection of the first vector onto the second.

    SOLUTION:

$$\cos(\theta) = \frac{1 + 0 + 3}{\sqrt{1 + 4 + 9}\sqrt{1 + 1}} = \frac{4}{\sqrt{28}} = \frac{2}{\sqrt{7}}$$

    Taking the inverse cosine give about 40.9 degrees. The projection gives the vector $\langle 2, 0, 2 \rangle$.

13. A constant force $\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$ moves an object along a line segment from $(1, 0, 2)$ to $(5, 3, 8)$. Find the work done (you may assume the distance is in meters, force is in newtons).

    SOLUTION: We need a direction vector: $4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ so that

$$W = \vec{F} \cdot \vec{D} = (12 + 15 + 60) \text{ J} = 87 \text{ J}$$

    (Note: I'm including the units of Joules to be precise, but wouldn't mark it down if you left it off- if you've had physics before, you really ought to include it!)

14. Graphical problems... We've mostly looked at these- We can go through some in class as well.

15. Do the lines below intersect? If so, find the point of intersection. If not, find the distance between them.

$$x = 3 + t, y = 2 - t, z = 1 \qquad x = 2 + s, y = 1 + 2s, z = 2 - s$$

SOLUTION: To check, we might note first that they are not parallel (the direction vectors being $\langle 1, -1, 0\rangle$ and $\langle 1, 2, -1\rangle$).

Any point of intersection will mean that, at some t and some s, the $x-$coordinates will be equal (and the same value of t and s would make the y and z coordinates equal as well).

From the equations for x, we have $t = s - 1$, so in the second equation,

$$2 - (s - 1) = 1 + 2s \quad \Rightarrow \quad s = \frac{2}{3} \quad \Rightarrow \quad t = -\frac{1}{3}$$

Substituting into the third equation, we see that either $z = 1$ or $z = 4/3$, therefore the lines do not intersect. Therefore, the two lines are skew.

Side Note: If the lines were parallel, to find the distance between them, take any point from one line, then find the distance from the **point** to the (other) line, which could be computed by using a projection.

To find the distance between the lines, we find parallel planes containing the two lines, then find the distance between them. The vector that is normal to the plane should be perpendicular to the two lines as well, so taking the cross product of the two direction vectors we get:

$$\vec{n} = \langle 1, -1, 0\rangle \times \langle 1, 2, -1\rangle = \langle 1, 1, 3\rangle$$

We need a point for each plane. For example, we might use $(3, 2, 1)$ and $(2, 1, 2)$, respectively. The (simplified) equations for the two planes are:

$$x + y + 3z - 8 = 0 \qquad x + y + 3z - 9 = 0$$

You can take a point on the first plane and find the distance to the second. For example, $(3, 2, 1)$ as we had before. Then the distance is:

$$\frac{|3 + 2 + 3(1) - 9|}{\sqrt{11}} = \frac{1}{\sqrt{11}}$$

16. Write the parametric equations for each situation:

(a) Go from $(1, 2)$ to $(-3, 2)$ as time runs from 0 to 1.
SOLUTION: Recall that if we want to go from a to b, then use $a(1 - t) + bt$

$$\begin{array}{lll} x & = 1(1-t) - 3t & = 1 - 4t \\ y & = 2(1-t) - 2t & = 2 \end{array}$$

(b) Go around the unit circle twice (clockwise) starting at $(-1, 0)$ as $0 \le t \le 1$.
SOLUTION: To go around the circle twice, the period of the cosine and sine will need to be 1/2- That means our basic set of equations will be:

$$\langle \cos(4\pi t), \sin(4\pi t) \rangle$$

To move clockwise, we can replace t by $-t$ (cosine is even, sine is odd):

$$\langle \cos(4\pi t), -\sin(4\pi t) \rangle$$

Next, shift time by π, so that at $t = 0$, we are using the angle π:

$$\langle \cos(4\pi t + \pi), -\sin(4\pi t + \pi) \rangle$$

(c) Parametrize the curve $y = 3x^2 + 5x + 2$:
SOLUTION: The simplest conversion is to take $x = t$:

$$\begin{aligned} x &= t \\ y &= 3t^2 + 5t + 2 \end{aligned}$$

(d) (**This was a bit confusing**- I wanted you to parametrize the polar curve, but I had a copy/paste error and asked you to plot it instead!).
Parametrize the curve $r = 1 + 2\cos(\theta)$ (in the Cartesian coordinate system) as $0 \le \theta \le 2\pi$.
SOLUTION: We know that $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so just substitute:

$$\begin{aligned} x &= (1 + 2\cos(\theta))\cos(\theta) \\ y &= (1 + 2\cos(\theta))\sin(\theta) \end{aligned}$$

17. Find the error (if there is one): Let $\mathbf{u}$ be a vector such that $|\mathbf{u}| = \sqrt{2}$. Choose a vector $\mathbf{v} \neq \mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{v} = 2$. Now we have:

$$\begin{aligned} 2(\mathbf{u} \cdot \mathbf{u}) &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} \\ 2(\mathbf{u} \cdot \mathbf{u}) - 2(\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - 2(\mathbf{u} \cdot \mathbf{v}) \\ 2(\mathbf{u} \cdot \mathbf{u}) - 2(\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} \\ 2\mathbf{u} \cdot (\mathbf{u} - \mathbf{v}) &= \mathbf{u} \cdot (\mathbf{u} - \mathbf{v}) \\ 2\mathbf{u} &= \mathbf{u} \\ 2 &= 1 \end{aligned}$$

SOLUTION:

We'll recall that $\mathbf{a} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{b}$ does NOT imply that $\mathbf{a} = \mathbf{c}$. In this case, you might note that the second, third and fourth lines are all of the form "zero=zero". Until the 5th line!

18. Show that $r = 2\cos(\theta)$ is a circle by converting it into Cartesian coordinates and writing it in standard form.

SOLUTION: Idea is not only the conversion, but also completing the square:

$$r = 2\cos(\theta) \quad \Rightarrow \quad r = \frac{2x}{r} \quad \Rightarrow \quad r^2 = 2x \quad \Rightarrow \quad x^2 - 2x + y^2 = 0 \quad \Rightarrow \quad (x-1)^2 + y^2 = 1$$

19. Find the scalar and vector projections of **a** onto **b**, if

- $\mathbf{a} = \langle -2, 3, -6 \rangle \qquad \mathbf{b} = \langle 5, -1, 4 \rangle$

  SOLUTION: The scalar projection is what the text refers to as

  $$\text{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{-10 - 3 - 24}{\sqrt{25 + 1 + 16}} = \frac{-37}{\sqrt{42}}$$

  The vector projection is then easy to compute:

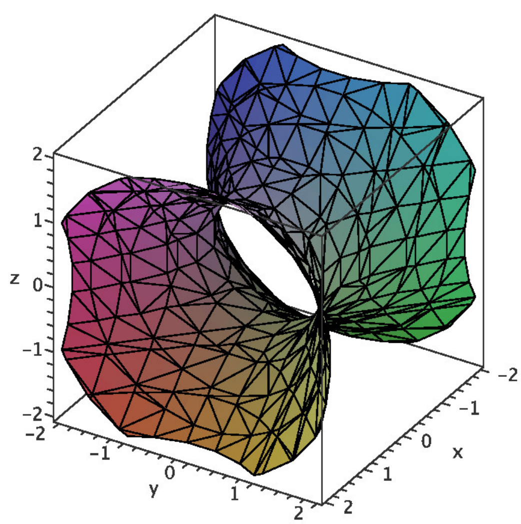
  $$\text{Proj}_{\mathbf{b}}(\mathbf{a}) = \frac{-37}{\sqrt{42}} \left(\frac{1}{\sqrt{42}} \right) \langle 5, -1, 4 \rangle = \frac{-37}{42} \langle 5, -1, 4 \rangle$$

- $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$

  SOLUTION: This one was included so that you would have practice with the standard basis vectors $\vec{i}, \vec{j}, \vec{k}$. In this case, the scalar projection is $-1/\sqrt{2}$ and the projection is $(-1/2)\mathbf{b} = (-1/2)\langle 0, 1, -1 \rangle$.

20. Sketch a few traces of the surface $y^2 + z^2 = 1 + x^2$ and describe the resulting surface (in words and/or as a sketch in three-dimensions).

SOLUTION:



21. Let a line L and a plane P be defined as:

$$L: \quad x = \frac{y+2}{3} = -z \qquad P_1: \quad x + y + z = 1$$

(a) The line in parametric form: $x = t \quad y = -2 + 3t \quad z = -t$

(b) Find the point of intersection between the plane and the line.

SOLUTION: Use the parametric form of the line, substitute it into the equation of the plane to get that $t = 1$. To check, substitute $t = 1$ into the equation of the line to get $(1, 1, -1)$. Substitute into the equation of a plane to see that the equation is satisfied.

(c) Given the point $Q(1, 2, 1)$, find the plane P_2 that contains the line L and the point Q.

SOLUTION: We have lots of points in the plane- We need the normal vector. We should see that the normal vector will be orthogonal to the line, or

$$\vec{n} \perp \langle 1, 3, -1 \rangle$$

Taking a vector from the line to the point (for example, from $(1, 2, 1)$ to $(0, -2, 0)$), we have that

$$\vec{n} \perp \langle 1, 4, 1 \rangle$$

Take the cross product to find that $\vec{n} = \langle 7, -2, 1 \rangle$, so one form of the plane is:

$$7(x-1) - 2(y-2) + (z-1) = 0$$

(d) Find the distance between Q and the plane P_1. What is the distance between planes P_1, P_2?

SOLUTION: Use the distance formula as before:

$$\frac{|1+2+1-1|}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Planes P_1 and P_2 are not parallel, so they will intersect (and the distance is therefore zero).

(e) Find the line of intersection between P_1, P_2.

SOLUTION: The two planes are $7(x-1)-2(y-2)+(z-1) = 0$ and $x+y+z = 1$. The line of intersection will have a direction orthogonal to both planes. Taking the cross product of the normals, the line has direction:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -3\mathbf{i} - 6\mathbf{j} + 9\mathbf{k}$$

Just to simplify things a bit, we'll use $\langle 1, 2, -3\rangle$ (this was not necessary, but it may be more convenient). Now we need a point of intersection. I recommend eliminating a variable to start with, for example:

$$\begin{array}{rl} 7x - 2y + z & = 4 \\ -(x + y + z & = 1) \\ \hline 6x - 3y \quad & = 3 \end{array}$$

Choose an x, y that works- For example, $x = 1, y = 1$. Then $z = -1$. Now we write the line in vector form as:

$$\langle 1, 1, -1\rangle + t\langle 1, 2, -3\rangle$$

22. Was it possible to let $\mathbf{u}$ be a vector such that $|\mathbf{u}| = 1$, and $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = 3$ with $|\mathbf{v}| = \sqrt{5}$?

    SOLUTION: No. Check the cosine formula for the dot product.

    $$\cos(\theta) = \frac{3}{\sqrt{5}} \approx 1.3416$$

    And there is no such θ, since the range of the cosine is between ± 1.

23. The volume of the parallelepiped is the absolute value of the scalar triple product, $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

    It doesn't matter which vector is which- if we cross $\mathbf{i} + \mathbf{j}$ with $\mathbf{j} + \mathbf{k}$, we get $\mathbf{i} - \mathbf{j} + \mathbf{k}$, and dotting that with the third vector gives the volume as 1 unit.

24. Write the vector $\langle -1, 5, 3\rangle$ as a sum of a vector *parallel* to $\langle 4, 2, 4\rangle$ and a vector *perpendicular* to $\langle 4, 2, 4\rangle$.

    SOLUTION: The vector parallel will be the (vector) projection:

    $$\frac{-4 + 10 + 12}{\sqrt{1 + 25 + 9}\sqrt{16 + 4 + 16}}\langle 4, 2, 4\rangle = \frac{3}{\sqrt{35}}\langle 4, 2, 4\rangle = \frac{6}{\sqrt{35}}\langle 2, 1, 2\rangle$$

    Lastly, the vector perpendicular will be the difference:

    $$\langle -1, 5, 3\rangle - \frac{6}{\sqrt{35}}\langle 2, 1, 2\rangle$$

    This doesn't simplify all that much, so OK to leave as this.