Exam 1 Sample Question SOLUTIONS

1. Eliminate the parameter to find a Cartesian equation for the curve: $x = \mathrm{e}^t$, $y = 2\mathrm{e}^t$.

   SOLUTION:

   You might look at the coordinates and notice that

$$y = 2x$$

   If you don't see it, we can go the long way:

$$\ln(x) = t \quad \Rightarrow \quad y = 2\mathrm{e}^{\ln(x)} = 2x$$

   Find the arc length of the curve from $t = 1$ to $t = 2$ of the original parametric form, then for the (x, y) form.

   SOLUTION 1 (Parametric form): In general form, the arc length is:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

   So in our case:

$$\int_1^2 \sqrt{\mathrm{e}^{2t} + 4\mathrm{e}^{2t}}\, dt = \int_1^2 \sqrt{5}\mathrm{e}^t\, dt = \sqrt{5}(\mathrm{e}^2 - \mathrm{e}^1)$$

   SOLUTION 2 (in the form $y = 2x$)

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{\mathrm{e}^1}^{\mathrm{e}^2} \sqrt{1 + 2^2}\, dx = \sqrt{5}(\mathrm{e}^2 - \mathrm{e}^1)$$

2. Find dy/dx and d^2y/dx^2, if $x = t^3 - 12t$ and $y = t^2 - 1$ (Hint: We do NOT want to try to convert it first).

   We recall that, using the Chain Rule:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2 - 12} = \frac{2t}{3(t^2 - 4)}$$

   and

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{2t}{3(t^2-4)}\right)}{3(t^2 - 4)} = \frac{-2}{9}\,\frac{t^2 + 4}{(t^2 - 4)^3}$$

3. Convert the polar equation to Cartesian:

$$r = \tan(\theta)\sec(\theta)$$

   SOLUTION: We recall that $\tan(\theta) = y/x$ and $\sec(\theta) = 1/cos(\theta)$, where $\cos(\theta) = x/r$. Substituting these, with $r = \sqrt{x^2 + y^2}$, we have:

$$\sqrt{x^2 + y^2} = \frac{y}{x} \cdot \frac{\sqrt{x^2 + y^2}}{x} \quad \Rightarrow \quad y = x^2$$

4. Convert the equation from Cartesian to polar of the form $r = f(\theta)$.

$$xy = 4$$

SOLUTION: Use our usual substitutions: $x = r\cos(\theta), y = r\sin(\theta)$:

$$r^2\cos(\theta)\sin(\theta) = 4 \quad \Rightarrow \quad r^2 = \frac{4}{\cos(\theta)\sin(\theta)}$$

5. Find the area of the surface obtained by rotating the curve about the $x-$axis:

$$x = 3t - t^3 \qquad y = 3t^2 \qquad 0 \le t \le 1$$

SOLUTIONS:

NOTE: You should think of this as a generic curve given in parametric form- You may assume it can be written as $y = f(x)$ (should have been given as a hint?)

$$\int_0^1 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2\pi\int_0^1 3t^2\sqrt{9t^4 + 18t^2 + 9}\,dt =$$

$$18\pi\int_0^1 t^2\sqrt{(t^2+1)^2}\,dt = 18\pi\int_0^1 t^2(t^2+1)\,dt = \frac{48}{5}\pi$$

6. Find the slope of the tangent line to the given polar curve at the point specified by θ:

$$r = 2 - \sin(\theta) \qquad \theta = \frac{\pi}{3}$$

SOLUTION:

$$x = r\cos(\theta) = (2 - \sin(\theta))\cos(\theta) = 2\cos(\theta) - \sin(\theta)\cos(\theta)$$

and

$$y = r\sin(\theta) = (2 - \sin(\theta)\sin(\theta) = 2\sin(\theta) - \sin^2(\theta)$$

After some algebra and trig,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2-\sqrt{3}}{1-2\sqrt{3}}$$

7. Show that the equation $r = a\sin(\theta) + b\cos(\theta)$, where $ab \neq 0$, represents the equation of a circle.

SOLUTION: Multiply through by r,

$$r^2 = ar\sin(\theta) + ar\cos(\theta) \quad \Rightarrow \quad x^2 + y^2 = ay + bx \quad \Rightarrow \quad x^2 - bx + y^2 - ay = 0$$

Complete the square in x and y:

$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2+b^2}{4}$$

NOTE: The restriction $ab \neq 0$ is more than we need (we just wanted that a, b are not both zero).

8. If we have two parallel planes, P_1 and P_2:

$$P_1: \quad ax + by + cz + d_1 = 0 \qquad P_2: \quad ax + by + cz + d_2 = 0$$

Then show that the distance between the planes is

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

SOLUTION: Recall that we pick a point off of one plane, then get the distance between that point and the other plane. For fun, let's take a point from Plane 2: (x_0, y_0, z_0). We note that, since this point is from plane 2, $ax_0 + by_0 + cz_0 + d_2 = 0$. Continuing, we use our distance formula:

$$\frac{|ax_0 + by_0 + cz_0 + d_1|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d_2 + d_1|}{\sqrt{a^2 + b^2 + c^2}}$$

9. Graphical problems: Like the back of Quiz 3 (online), prob. 24, 25, 26 on p. 627.

10. Do the lines below intersect? If so, find the point of intersection. If not, find the distance between them.

$$x = 3 + t, y = 2 - t, z = 1 \qquad x = 2 + s, y = 1 + 2s, z = 2 - s$$

SOLUTION: To see if the lines intersect, set the coordinates equal to each other:

$$3 + t = 2 + s \qquad 2 - t = 1 + 2s \qquad 1 = 2 - s$$

Choose any two to solve for t, s. Here, we choose the first and third equations and get:

$$s = 1 \quad t = 0$$

We see that in this case, the x coordinates match ($x = 3$) and the $z-$coordinates match ($z = 1$), but the y coordinates do not match ($y = 1$ versus $y = 3$). Therefore the lines are skew, and we will find the distance between them.

In class, we learned that we can do this by looking at the two lines as being in two parallel planes, then find the distance between the planes (we might use our newly found formula!).

We have a point on each plane ($(3, 2, 1)$ and $(3, 3, 1)$)- To find the normal vector, we take the cross product of the directions of each line:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 2 & -1 \end{vmatrix} = < 1, 1, 3 >$$

If you'd like, you can write the equations of the two planes out:

$$\text{Plane 1: } (x - 3) + (y - 2) + 3(z - 1) = 0 \qquad x + y + 3z - 8 = 0$$

Plane 2: $(x-3)+(y-3)+3(z-1)=0 \qquad x+y+3z-9=0$

From the distance formula we got earlier, the distance is:

$$\frac{1}{\sqrt{11}}$$

11. Find an equation for the surface obtained by rotating the parabola $y = x^2$ about the $y-$axis. (Hint: In 3-d, if you fix a $y-$value, what shape should you have in the $xz-$plane?)

    SOLUTION: We can think of the graph $y = x^2$ as the result of plotting the trace when $z = 0$. If we rotate the curve about the $y-$axis, we'll get something like a rounded vase- If we take a cross section parallel to the $xz-$plane when $y = k$, we should get a circle of radius $\sqrt{k}$, so our equation is:

    $$x^2 + z^2 = y$$

12. Find the distance between the origin and the line

    $$x = 1 + t \quad y = 2 - t \quad z = -1 + 2t$$

    Hint: Take an arbitrary point on the line and form two vectors so that the distance can be found- perhaps with a sine? Hint 2: It is possible to solve this with Calculus- Use it to check your answer.

    SOLUTION: One point on the line can be found by letting $t = 0$ and we have $P(1, 2, -1)$, and let $Q(0, 0, 0)$. Now we have a triangle between $\vec{PQ}$, $\mathbf{b} =< 1, -1, 2 >$ and the vector going formed by taking the projection of $\vec{PQ}$- Its easiest to draw a picture and you'll find that, if h is the length we're looking for, then

    $$\sin(\theta) = \frac{h}{|\vec{PQ}|} \qquad \Rightarrow \qquad h = \frac{|\vec{PQ} \times b|}{|\mathbf{b}|} = \sqrt{\frac{3^3}{6}} = \frac{3}{\sqrt{2}}$$

    For fun, we can do this with calculus using the distance squared (you might remember this when we used to find the maximum/minimum back in Calc I):

    $$h = (1 + t - 0)^2 + (2 - t - 0)^2 + (-1 + 2t - 0)^2 = 6 - 6t + 6t^2$$

    The minimum occurs at $t = 1/2$ (find by setting the derivative to zero), which gives $9/2$ (the distance squared).

13. Given the property:

    $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

    If vectors $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, and

    $$\mathbf{c} = \mathbf{a} \times (\mathbf{a} \times \mathbf{b})$$

- Is $\mathbf{c}$ a unit vector?

  That doesn't necessarily follow- If we look at the algebra we have

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b} = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - \mathbf{b}$$

  Now, the magnitude (squared) is the dot product:

$$|\mathbf{c}|^2 = ((\mathbf{a}\cdot\mathbf{b})\mathbf{a}-\mathbf{b})\cdot((\mathbf{a}\cdot\mathbf{b})\mathbf{a}-\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{a})-2(\mathbf{a}\cdot\mathbf{b})^2+(\mathbf{b}\cdot\mathbf{b}) = \cos(\theta)-2\cos^2(\theta)+1$$

  So the magnitude depends on the angle between them. Notice that, if the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, then $\mathbf{c} = -\mathbf{b}$, and it would have unit length.

- Is $\mathbf{c} \perp \mathbf{a}$?

  Yes. The resulting vector of the cross product is orthogonal to the vectors.

- Is $\mathbf{c} \perp \mathbf{b}$?

  Not necessarily. For example, we saw that if $\mathbf{a} \perp \mathbf{b}$, then $\mathbf{c} = -\mathbf{b}$, and they would be parallel. We could also pull out a numerical example:

$$\mathbf{a} = \frac{1}{\sqrt{2}} < 1, 1, 0 > \qquad \mathbf{b} = \frac{1}{\sqrt{3}} < 1, 1, 1 >$$

$$\mathbf{a} \times \mathbf{b} = \frac{1}{\sqrt{6}} < 1, -1, 0 > \qquad \mathbf{c} = \frac{1}{\sqrt{3}} < 0, 0, -1 >$$

14. Assume that $\mathbf{a} \neq \mathbf{0}$. Explain your answer (if "No", provide an example):

   (a) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

   No- This says, in essence, that if two projections are the same, then the vectors were the same and that is false. For example,

$$\mathbf{a} =< 1, 0 > \quad \mathbf{b} =< 1, 1 > \quad \mathbf{c} =< 1, 3 >$$

   (b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

   Not necessarily. Think of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ all in the same plane, so that $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$ would both give you the normal to the plane.

   (c) If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$?

   Yes. Let's see- The dot product implies:

$$\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} = \mathbf{0} \quad \Rightarrow \quad \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{0} \quad \Rightarrow \quad \mathbf{a} \perp (\mathbf{b} - \mathbf{c})$$

   The cross product (re-written to look more like the rules in the table):

$$\mathbf{c} \times \mathbf{a} - \mathbf{b} \times \mathbf{a} = (\mathbf{c} - \mathbf{b}) \times \mathbf{a} = \mathbf{0} \quad \Rightarrow \quad \mathbf{a} \parallel (\mathbf{b} - \mathbf{c})$$

   Can a vector $\mathbf{a}$ be both parallel and perpendicular to the same vector? Only if that vector is zero. Therefore, $\mathbf{b} - \mathbf{c} = \mathbf{0}$ and therefore $\mathbf{b} = \mathbf{c}$.