Exam 2 Sample Questions

Be sure to look over your old quizzes and homework as well. Here are the formulas that I will give you on the exam:

Formulas for Curvature

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \text{ or } \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$$

Formulas for components of acceleration in 3d

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} \qquad a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

1. True or False, and explain:

   (a) There exists a function f with continuous second partial derivatives such that
   $$f_x(x,y) = x + y^2 \qquad f_y = x - y^2$$

   (b) The function f below is continuous at the origin.
   $$f(x,y) = \begin{cases} \frac{2xy}{x^2+2y^2}, & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

   (c) If $\vec{r}(t)$ is a differentiable vector function, then
   $$\frac{d}{dt}|\vec{r}(t)| = |\vec{r}\,'(t)|$$

   (d) If $z = 1 - x^2 - y^2$, then the linearization of z at $(1,1)$ is
   $$L(x,y) = -2x(x-1) - 2y(y-1)$$

   (e) We can always use the theorem: $\nabla f(a,b)\cdot\vec{u}$ to compute the directional derivative at (a,b) in the direction of $\vec{u}$.

   (f) Different parameterizations of the same curve result in identical tangent vectors at a given point on the curve.

   (g) If $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector functions, then
   $$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}\,'(t)\times\vec{v}\,'(t)$$

   (h) If $f_x(a,b)$ and $f_y(a,b)$ both exist, then f is differentiable at (a,b).

   (i) At a given point on a curve $(x(t_0), y(t_0), z_0(t))$, the osculating plane through that point is the plane through $(x(t_0), y(t_0), z(t_0))$ with normal vector is $\vec{B}(t_0)$.

2. Show that, if $|\vec{r}(t)|$ is a constant, then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$. (HINT: Differentiate $|\vec{r}(t)|^2 = k$)

3. Is it possible for the directional derivative to exist for every unit vector $\vec{u}$ at some point (a,b), but f is still not differentiable there?

   Consider the function $f(x,y) = \sqrt[3]{x^2y}$. Show that the directional derivative exists at the origin (by letting $\vec{u} = \langle\cos(\theta),\sin(\theta)\rangle$ and using the **definition**), BUT, f is not differentiable at the origin (because if it were, we could use $\nabla f\cdot\vec{u}$ to compute $D_{\vec{u}}f$).

4. If $f(x,y) = \sin(2x+3y)$, then find the linearization of f at $(-3,2)$.

5. The radius of a right circular cone is increasing at a rate of 3.5 inches per second while its height is decreasing at a rate of 4.3 inches per second. At what rate is the volume changing when the radius is 160 inches and the height is 200 inches? ($V = \frac{1}{3}\pi r^2 h$)

6. Find the differential of the function: $v = y\cos(xy)$

7. Find the maximum rate of change of $f(x,y) = x^2y + \sqrt{y}$ at the point $(2,1)$, and the direction in which it occurs.

8. Find an expression for

$$\frac{d}{dt}\left[\mathbf{u}(t)\cdot(\mathbf{v}(t)\times\mathbf{w}(t))\right]$$

9. Use Lagrange Multipliers to find the maximum and minimum of f subject to the given constraints:

$$f(x,y) = x^2y \qquad x^2+y^2=1$$

10. The curves below intersect at the origin. Find the angle of intersection to the nearest degree:

$$\vec{r}_1(t) = \langle t, t^2, t^9\rangle \quad \vec{r}_2(t) = \langle \sin(t), \sin(5t), t\rangle$$

11. Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

12. Find the equation of the tangent plane and normal line to the given surface at the specified point:

$$x^2+2y^2-3z^2=3 \qquad (2,-1,1)$$

13. If $z = x^2 - y^2$, $x = w + 4t$, $y = w^2 - 5t + 4$, $w = r^2 - 5u$, $t = 3r + 5u$, find $\partial z/\partial r$.

14. If $x^2 + y^2 + z^2 = 3xyz$ and we treat z as an implicit function of z, then find $\partial z/\partial x$ and $\partial z/\partial y$.

15. Short Answer:

    (a) If the velocity is $\sin(t)\vec{i} - \cos(t)\vec{j} + 14t\vec{k}$, and the initial position is $\vec{i}+\vec{j}+2\vec{k}$, (i) Find the position function, (ii) Find the acceleration, (iii) Find the tangential and normal components of acceleration.

    (b) Find the length of the curve:

$$\mathbf{r}(t) = \langle 9\sin(t), 3t, 9\cos(t)\rangle, -\pi \le t \le 2\pi$$

16. At what point does $y = 8e^x$ have maximum curvature?

17. Find the equation of the normal line through the level curve $4 = \sqrt{5x-4y}$ at $(4,1)$ using a gradient.

18. Find all points at which the direction of fastest change in the function $f(x,y) = x^2+y^2-2x-4y$ is $\vec{i}+\vec{j}$.

19. Find and classify the critical points:

$$f(x,y) = 4 + x^3 + y^3 - 3xy$$