The Gradient and the Level Curve

Our text does not show this, but the fact that the gradient is orthogonal to the level curve comes up again and again, and in fact, the text proves a more complicated version in three dimensions (the gradient is orthogonal to the level *surface*). It is important, so we go through a proof and an example.

Set up: We'll set things up very carefully so that the question becomes clearer. We'll also follow along each step with a specific example.

- Let $z = f(x, y)$ be our function (graphically a surface), and let (a, b) be in the domain of f.

  Example: $z = x^2 + y^2$ at $(1, 3)$.

- Fix the level curve $k = f(x, y)$ by taking $k = f(a, b)$.

  Example: The level curve is at the level $k = 1^2 + 3^2 = 10$, or $x^2 + y^2 = 10$ (Notice that we have guaranteed that the point is on this level curve).

Now consider the gradient vector at the point (a, b), which we denote by

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle$$

We want to show that this two dimensional vector is orthogonal to the level curve at (a, b) (which means the vector is orthogonal to the tangent line at (a, b)) so we need to find the direction in which the tangent line goes.

An alternative would be to find the slope of the line through (a, b) in the direction of the gradient (our book calls this the normal line). If this line is perpendicular to our tangent line, then the slopes ought to be negative reciprocals of each other.

Example: The gradient is $\nabla f = \langle 2x, 2y\rangle$, so at $(1, 3)$, the gradient is $\nabla f(1, 3) = \langle 2, 6\rangle$.

We need to show that $\langle 2, 6\rangle$ is orthogonal to the tangent line for $x^2 + y^2 = 10$ at $(1, 3)$. We'll find the slope of the tangent line next.

We know how to differentiate $f(x, y) = k$ implicitly (taking y as a function of x):

$$f_x(x, y) + f_y(x, y)\frac{dy}{dx} = 0$$

Therefore, the slope of the tangent line at (a, b) will be:

$$\frac{dy}{dx} = \frac{-f_x(a, b)}{f_y(a, b)}$$

Alternatively, the slope of the line in the direction of the gradient (based at the point (a, b)) is:

$$m = \frac{\Delta y}{\Delta x} = \frac{f_y(a, b)}{f_x(a, b)}$$

Example: To finish off our example, find the equations of the tangent line and normal line. For the tangent line,

$$x^2 + y^2 = 10 \quad \Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \quad \text{at } (1,3) \quad \Rightarrow \frac{dy}{dx} = -\frac{1}{3}$$

And for the normal line, we go through the point $(1,3)$ in the direction of the gradient $\langle 2, 6\rangle$, so the slope is

$$m = \frac{6}{2} = 3$$

And we see that the gradient is indeed orthogonal to the level curve. For fun, finish the equations of the lines and plot everything- The tangent and normal lines would be

$$y - 3 = -\frac{1}{3}(x-1) \qquad y - 3 = 3(x-1)$$