Even numbered problems and the "story problems": Section 14.5

12. Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$:

$$z = \tan(u/v) \qquad u = 2s + 3t \qquad v = 3s - 2t$$

SOLUTION: You should draw a tree diagram to help keep your variables straight. We'll need all the partial derivatives, so we'll go ahead and compute them all:

$z = \tan(u/v)$	$u = 2s + 3t$	$v = 3s - 2t$
$z_u = \sec^2(u/v) \cdot (1/v)$	$u_s = 2$	$v_s = 3$
$z_v = \sec^2(u/v) \cdot (-u/v^2)$	$u_t = 3$	$v_t = -2$

Now, write out what we need, substitute and simplify a bit:

$$\frac{\partial z}{\partial s} = z_u \cdot u_s + z_v \cdot v_s = \frac{2\sec^2(u/v)}{v} + \frac{-3u\sec^2(u/v)}{v^2} = \left(\frac{2v - 3u}{v^2}\right)\sec^2(u/v)$$

And for the other one:

$$\frac{\partial z}{\partial t} = z_u \cdot u_t + z_v \cdot v_t = \frac{3\sec^2(u/v)}{v} + \frac{2u\sec^2(u/v)}{v^2} = \left(\frac{3v + 2u}{v^2}\right)\sec^2(u/v)$$

Side Remark: You might notice that these partial derivatives are written in u, v and not in s, t- As it happened, s, t ended up not appearing at all! It is OK to leave in this form unless you're asked to back substitute to get expressions in s and t (by using $u = 2s + 3t$ and $v = 3s - 2t$)

14. Let $W = F(u, v)$ where u is a function of s, t and v is a function of s, t. We also have the table of values:

$u(1,0) = 2$	$v(1,0) = 3$	
$u_s(1,0) = -2$	$v_s(1,0) = 5$	$F_u(2,3) = -1$
$u_t(1,0) = 6$	$v_t(1,0) = 4$	$F_v(2,3) = 10$

Find $W_s(1,0)$ and $W_t(1,0)$.

SOLUTION: Nice problem! Go directly from our notation into the numbers. First,

$$W_s = F_u\, u_s + F_v\, v_s \quad \Rightarrow \quad W_s(1,0) = (-1)(-2) + (10)(5) = 52$$

And

$$W_t = F_u\, u_t + F_v\, v_t \quad \Rightarrow \quad W_t(1,0) = (-1)(6) + (10)(4) = 34$$

35. The temperature at a point (x, y) is $T(x, y)$ measured in degrees Celsius. A bug crawls so taht its position after t seconds is

$$x = \sqrt{1+t} \qquad y = 2 + \frac{1}{3}t$$

where x, y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$.

How fast is the temperature rising on the bug's path after 3 seconds?

SOLUTION: We should interpret the question to mean: Find dT/dt when $t = 3$, which means $x = \sqrt{4} = 2$ and $y = 2 + 1 = 3$:

$$\frac{dT}{dt} = T_x(2, 3)\, x'(3) + T_y(2, 3)\, y'(3)$$

Side note: In this case, x and y were functions of one variable only, so the notation x' is meaningful here, and it means to differentiate in time.

We only need $x'(t) = \frac{1}{2}(1+t)^{-1/2}$, so $x'(3) = \frac{1}{4}$ and $y'(t) = \frac{1}{3}$. Substitute:

$$\frac{dT}{dt} = 4 \cdot \frac{1}{4} + 3 \cdot \frac{1}{3} = 2$$

In terms of the bug, the temperature is rising at an (instantaneous) rate of 2°C per second at $t = 3$.

38. The radius of a right circular cone is increasing at a rate of 1.8 inches per second, while its height is decreasing at a rate of 2.5 inches per second. At what rate is the volume of the cone changing when the radius is 120 inches and the height is 140 inches.

SOLUTION: The volume of the cone (see the front notes of the book) is:

$$V(r, h) = \frac{1}{3}\pi r^2 h$$

Where we think of r and h as functions of time (so then volume is also a function of time). Therefore,

$$\frac{dV}{dt} = V_r \cdot \frac{dr}{dt} + V_h \frac{dh}{dt}$$

We compute each of these quantities and then substitute:

$$\begin{array}{r|l|} V_r = (2/3)\pi r h & V_r(120, 140) = 11,200\pi \\ V_h = (1/3)\pi r^2 & V_h(120, 140) = 4,800\pi \end{array}$$

$$\frac{dV}{dt} = 11200\pi \cdot 1.8 + 4800\pi(-2.5) = 8160\pi \text{ in}^3/\text{sec}$$

39. A box is given, and at a certain instant the dimensions are $l = 1$, and $w = h = 2$. Quantities l, w are decreasing at a rate of 2 meters per second (everything is in meters), while h is decreasing at a rate of 3 meters per second. At that instant find the rates at which the following are changing:

(a) Volume
SOLUTION: The relationship is $V = lwh$, where l, w, h are functions of time:

$$\frac{dV}{dt} = V_l\frac{dl}{dt} + V_w\frac{dw}{dt} + V_h\frac{dh}{dt} = whl' + lhw' + lwh' = 6$$

(b) Surface Area:
SOLUTION: $A = 2(lw + lh + wh)$, and

$$\frac{dA}{dt} = A_l l' + A_w w' + A_h h' = 10$$

(c) The length of a diagonal (HINT: It is easier if we think of each of these as functions of time):

$$L(t)^2 = l(t)^2 + w(t)^2 + h(t)^2$$

We want dL/dt:

$$2L\frac{dL}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt} + 2h\frac{dh}{dt} = 0$$

48. If $z = f(x, y)$, $x = s + t$ and $y = s - t$, show that

$$z_x^2 - z_y^2 = z_s z_t$$

Compute the partials of z with respect to s and t, then simplify the right-hand side of the equation:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = z_x + z_y$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = z_x - z_y$$

Therefore, $z_s z_t = (z_x + z_y)(z_x - z_y) = z_x^2 - z_y^2$

49. Show that any function of the form

$$z = f(x - at) + g(x + at)$$

is a solution of the "wave equation",

$$z_{tt}^2 = a^2 z_{xx}^2$$

Hint: Let $u = x + at, v = x - at$.

SOLUTION: Following the hint, we take z to be a function of u and v,

$$z = f(u) + g(v), \qquad u = x + at \quad v = x - at$$

Therefore, treat f, g as functions of u, v, and differentiate using the chain rule. We recall that

$$z_{tt} = f_{tt} + g_{tt} \qquad z_{xx} = f_{xx} + g_{xx}$$

so we'll compute these separately and put them together at the end. We'll also go ahead and compute the partials of u and v:

$$u_t = a \quad u_x = 1 \quad v_t = -a \quad v_x = 1$$

First, the derivatives with respect to t:

$$f_t = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} = f_u(a) + (0)(a) = af_u$$

Where $f_v = 0$ since f is a function of u only. Differentiating again,

$$f_{tt} = (af_u)_t = a\left(\frac{\partial f_u}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f_u}{\partial v}\frac{\partial v}{\partial t}\right) = a(f_{uu}(a) + 0) = a^2 f_{uu}$$

Do the same thing with $g(v)$ (now $g_u = 0$):

$$g_t = \frac{\partial g}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial t} = 0 + g_v(-a) = -ag_v$$

Differentiating again,

$$g_{tt} = (-ag_v)_t = -a\left(\frac{\partial g_v}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial g_v}{\partial v}\frac{\partial v}{\partial t}\right) = -a(0 - ag_{vv}) = a^2 g_{vv}$$

So far we have shown that:

$$z_{tt} = f_{tt} + g_{tt} = a^2(f_{uu} + g_{vv})$$

To finish this exercise, show that $a^2 z_{xx}$ is actually the right hand side of the equation above. Start with f_x, and g_x then compute the second derivatives (its gets faster with practice!). Here is some shortcut notation that makes it a bit easier:

$$f_x = f_u u_x + f_v v_x = f_u + 0 = f_u$$

$$f_{xx} = (f_u)_x = (f_u)_u u_x + (f_u)_v v_x = f_{uu} + 0 = f_{uu}$$

And

$$g_x = g_u u_x + g_v v_x = 0 + g_v = g_v$$

$$g_{xx} = (g_v)_x = (g_v)_u u_x + (g_v)_v v_x = 0 + g_{vv} = g_{vv}$$

Therefore, $z_{xx} = f_{uu} + g_{vv}$, which is what we wanted.