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Definition

A function $z = f(x, y)$ has a global min (max) at a point (a, b) in a given region D if $f(a, b)$ is the smallest (largest) point in all of D (could be equality, too- There could be multiple max's and min's).

As in Calc I, we have the Extreme Value Theorem:

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Definition

The critical points of $z = f(x, y)$ are points where $\nabla f = 0$ or either (or both) partial derivatives do not exist.

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In the case that the EVT applies (global max/min on a closed and bounded domain), the candidates for where the max/min can occur:

- Critical points
- Boundary

Check them, and find the max/min on each (build a table).

Example: Find the global max and global min:

$$f(x, y) = 5 + x^2 + x - 2y^2 \qquad -1 \leq x \leq 1, \quad -1 \leq y \leq 1$$

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SOLUTION: Find critical points:

$$f_x(x,y) = 2x + 1 \qquad f_y(x,y) = -4y \quad \Rightarrow \quad (-1/2, 0)$$

Value of f at the critical point: 4.75.

Check the boundary:

- $f(x, y) = 5 + x^2 + x - 2y^2$ for $x = 1$, $-1 \leq y \leq 1$:

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y	$f(1,y)$
-1	$f(1,-1) = 5$
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-1	$f(1, -1) = 5$
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- $x = -1$, $-1 \leq y \leq 1$:

$$f(-1, y) = 5 - 2y^2 \quad -1 \leq y \leq 1$$

y	$f(1, y)$
-1	$f(-1, -1) = 3$
0	$f(-1, 0) = 5$
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x	$f(x,-1)$
-1	$f(-1,-1) = 3$
$-1/2$	$f(-1/2,-1) = 2.75$
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$-1/2$	$f(-1/2, -1) = 2.75$
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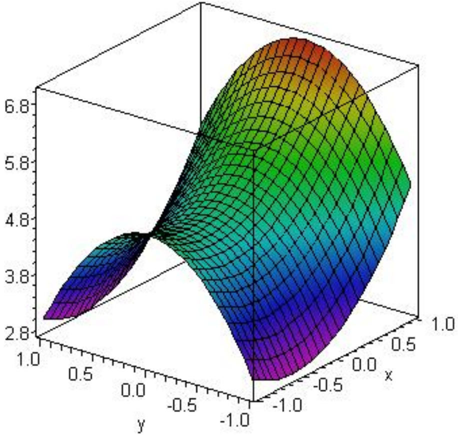
- For $y = -1$, we have the same function and same interval.

Conclusion:

The global maximum is 7, it occurs at $(1, 0)$ on the boundary. The global minimum is 2.75, it occurs twice on the boundary, at $(-1/2, \pm 1)$.



Local Extrema

To find local extrema, in Calc I we had the first and second derivative tests. It is not easy to find a substitute- A surface can be both CU and CD at a *saddle point*.

The Second Derivatives Test

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- If $D = 0$, the test fails (we could have local max, local min or saddle).

Example: Classify the Critical Points

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$$f_{xx} = 1 \quad f_{xy} = -3 \quad f_{yy} = 18y + 18$$

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Therefore, $y = -2$ and $y = 1$. Backsub to get the ordered pairs:

$$(3, -2) \qquad (12, 1)$$

Do the Second Derivatives test on each CP; simplify first:

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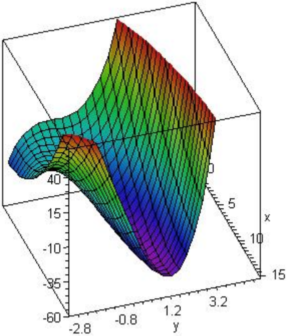
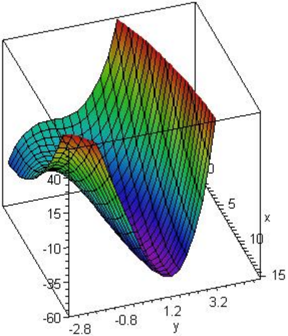

40
15
-10
-35
-60
-2.8
-0.8
1.2
3.2
y
5
x
10
15


40
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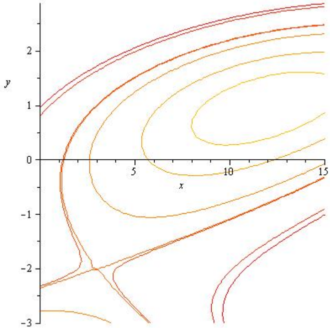
2
1
0
-1
-2
-3
y
5
10
15
x

Example

Find the local maximum, minimum and saddle points. Verify your answer by locating these points on the plot of level curves.

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$$g_x = y(1 - 2x - y) \qquad g_{xx} = -2y \qquad g_{xy} = 1 - 2x - 2y$$
$$g_y = x(1 - x - 2y) \qquad g_{yy} = -2x$$

Solving for the critical points,

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Now we have two more fixed points: $(0,1)$ or $(1/3,1/3)$.

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$(0,1)$	-1	N/A	Saddle
$(1/3,1/3)$	$1/3$	$-2/3$	

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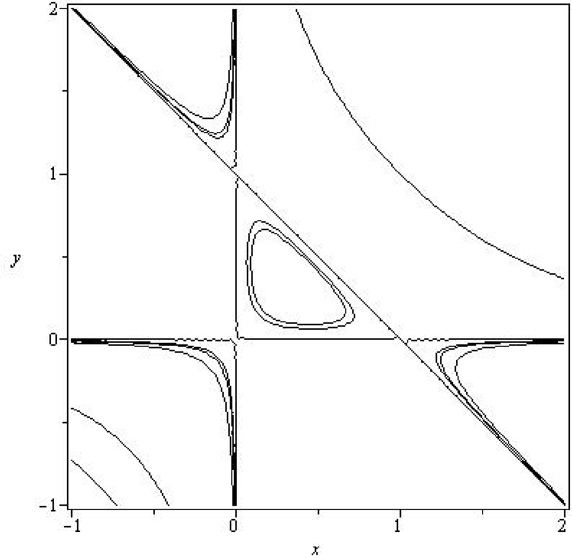
$(1/3,1/3)$	$1/3$	$-2/3$	Local Max

Here is the contour plot, and we see the saddles and local max:



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and second derivatives:

$$f_{xx} = (2y - 10x^2 y + 4x^4 y)e^{-x^2-y^2} \qquad f_{yy} = (4x^2y^3 - 6x^2 y)e^{-x^2-y^2}$$

and

$$f_{xy} = 2x(1 - x^2 - 2y^2 + 2x^2y^2)e^{-x^2-y^2}$$

Put everything in a table:

Point	D	f_{xx}	Result
$(0, y)$	0	0	

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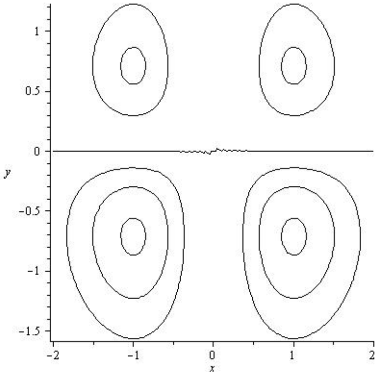
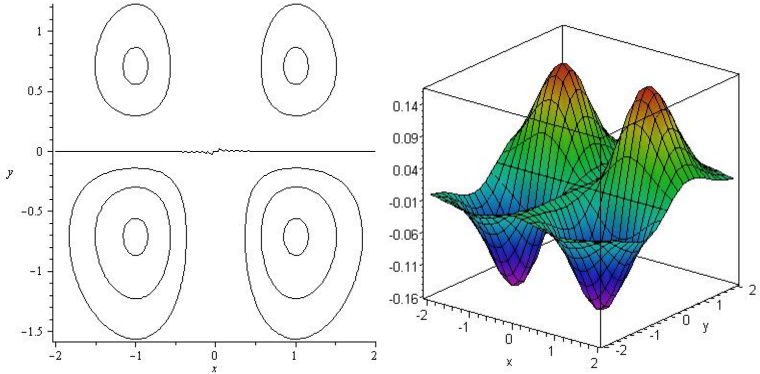


1
0.5
0
y
-0.5
-1
-1.5
-2
-1
0
1
2
x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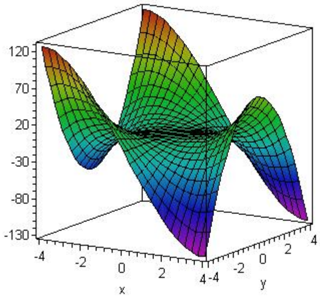
0
y
-0.5
-1
-1.5
-2
-1
0
1
2
x
0.14
0.09
0.04
-0.01
-0.06
-0.11
-0.16
-2
-1
0
1
2
x
2
1
0
y
-1
-2

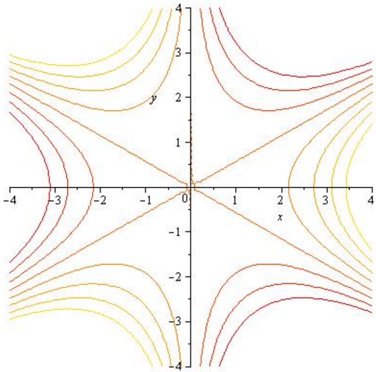
From the graph, we see that if $y > 0$, then points $(0, y)$ are where local minima occur, and if $y > 0$, then $(0, y)$ are where local maxima occur. These would be difficult to determine without the graph.

If $D = 0$, some complicated behaviors can occur. In this example, we have

$$f(x, y) = x^3 - 3xy^2$$

Below is the surface, called a "Monkey Saddle", and the corresponding contour plot.


120
70
20
-30
-80
-130
-4
-2
0
2
4
x
-4
-2
0
2
4
y


4
3
2
1
-1
-2
-3
-4
y
-4
-3
-2
-1
0
1
2
3
4
x