Selected Solutions, Sections 16.6 and 7

16.6, 37 Find the area of the surface for the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

SOLUTION: If the surface is defined as $\vec{r}(x, y)$, then integrating the area of one patch: $|\vec{r}_x \times \vec{r}_y|$ over the whole domain D will give the surface area. In this case we can write

$$\vec{r}(x,y) = \langle x, y, 6 - 3x - 2y \rangle \quad \Rightarrow \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{(-3)^2 + (-2)^2 + 1}$$

$$\iint_S dS = \iint_D \sqrt{14}\, dA =$$

To see where the plane intersects the first octant, look for the intercepts with the x, y and z axes:

$$(2,0,0), (0,3,0), (0,0,6)$$

So the domain D is the triangle in the first quadrant of the xy plane bounded by $(0,0), (2,0)$ and $(0,3)$. We could compute this using geometry:

$$\iint_D \sqrt{14}\, dA = \sqrt{14}\frac{1}{2}(2)(3) = 3\sqrt{14}$$

16.6, 39 In this case, the surface is defined as $z = f(x,y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ over the square $0 \le x \le 1$, $0 \le y \le 1$.

The surface area element that we will integrate is:

$$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{(x^{1/2})^2 + (y^{1/2})^2 + 1} = \sqrt{1 + x + y}$$

The surface area is:

$$\int_0^1 \int_0^1 \sqrt{1 + x + y}\, dy\, dx$$

Let $u = 1 + x + y$ so that $du = dy$ (treat x as constant). If $y = 0, u = x + 1$ and if $y = 1$, $u = x + 2$:

$$\int_0^1 \frac{2}{3} u^{3/2} \Big|_{x+1}^{x+2} dx = \frac{2}{3}\int_0^1 (x+2)^{3/2} - (x+1)^{3/2}\, dx = \frac{2}{3} \cdot \frac{2}{5}\left((x+2)^{5/2} - (x+1)^{5/2}\Big|_0^1\right.$$

Evaluating this last term gies us $\frac{4}{15}(3^{5/2} - 2^{7/2} + 1) \approx 1.407$

16.6, 41 The surface area will be:

$$\iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA = \iint_D \sqrt{y^2 + x^2 + 1}\, dA$$

Convert to polar, and let $u = 1 + r^2$, so that $du = 2r\, dr$:

$$\int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}\, r\, dr\, d\theta = 2\pi \cdot \frac{1}{2}\int_1^2 u^{1/2}\, du = 2\pi \cdot \frac{1}{2} \cdot \frac{2}{3} u^{3/2}\Big|_1^2 = \frac{2\pi}{3}(2^{3/2} - 1)$$

16.7, 5 To compute this surface integral, we notice that the surface is given as the plane $z = 1 + 2x + 3y$. Therefore,

$$dS = \sqrt{f_x^2 + f_y^2 + 1}\, dA = \sqrt{2^2 + 3^2 + 1}\, dA$$

The integrand is x^2yz which, when its domain is restricted to the surface S will be: $x^2y(1 + 2x + 3y)$. Therefore,

$$\iint_S x^2yz\, dS = \int_0^3 \int_0^2 x^2y(1 + 2x + 3y)\sqrt{14}\, dy\, dx$$

Simplify and integrate:

$$\sqrt{14}\int_0^3 \int_0^2 x^2y + 2x^3y + 3x^2y^2\, dy\, dx = \sqrt{14}\int_0^3 10x^2 + 4x^3\, dx = 171\sqrt{14} \approx 639.82$$

16.7, 7 This one has the same idea as Exercise 5: In this case, our surface area term is:

$$dS = \sqrt{1^2 + 1^2 + 1}\, dA = \sqrt{3}\, dA$$

And the integrand is yz- It is restricted to the surface S, so it becomes $y(1 - x - y)$.

To find the domain D in the $xy-$plane, if $z = 0$, the line is $x + y = 1$ (just in the first quadrant). Therefore,

$$\iint_S yz\, dS = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} y(1 - x - y)\sqrt{3}\, dy\, dx = \cdots = \frac{\sqrt{3}}{24}$$

16.7, 13 **You do not need to be responsible for this problem.**

It is set up to be parameterized in x, z instead of our usual x, y. If you're curious about it, here is the solution:

$$\vec{r}(x, z) = \langle x, x^2 + z^2, z\rangle \quad \Rightarrow \quad \vec{r}_x = \langle 1, 2x, 0\rangle, \quad \vec{r}_z = \langle 0, 2z, 1\rangle$$

so that

$$|\vec{r}_x \times \vec{r}_z| = \sqrt{4x^2 + 1 + 4z^2} = \sqrt{1 + 4(x^2 + z^2)}$$

Therefore,

$$\iint_S y\, dS = \iint_{x^2+z^2 \le 4} (x^2 + z^2)\sqrt{1 + 4(x^2 + z^2)}\, dA$$

Using $x = r\cos(\theta)$ and $z = r\sin(\theta)$, we convert to polar coordinates and proceed as usual.

NOTE: You could just swap y and z in the statement of the problem, and then everything would look normal.

16.7, 17 **You do not need to be responsible for this problem.**

(The surface parameterization is done differently than what we will be expecting for the exam)

16.7, 19 In this case, the surface S is parameterized the way we expect, $z = 4 - x^2 - y^2$. To set up the flux,

$$\vec{F} = \langle xy, yz, zx \rangle = \langle xy, y(4 - x^2 - y^2), x(4 - x^2 - y^2) \rangle$$

Think about this: Those two vectors for $\vec{F}$ do not look like they are the same. How is it that we put an equality there?

ANSWER: It is because we are restricting the domain of $\vec{F}$ to be only on the surface S (defined by $z = f(x, y)$), and NOT all of $\mathbb{R}^3$.

Set up the flux:

$$\vec{F} \cdot d\vec{S} = \langle xy, y(4 - x^2 - y^2), x(4 - x^2 - y^2) \rangle \cdot \langle -(-2x), -(-2y), 1 \rangle \, dA =$$

The integral is:

$$\int_0^1 \int_0^1 2x^2y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2) \, dy \, dx =$$

$$\int_0^1 \frac{34}{35} + \frac{1}{3}x^2 + \frac{11}{3}x - x^3 \, dx = \frac{713}{180} \approx 3.961$$

16.7, 21 In this case, the surface S is parameterized as

$$\vec{r}(x, y) = \langle x, y, 1 - x - y \rangle$$

We want *downward* orientation. In our usual setup, the normal vector is outward and upward:

$$\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle = \langle 1, 1, 1 \rangle$$

So we can either take the negative of this, or take the negative of the integral. We'll just take the negative at the end.

Continuing, take:

$$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = \langle x(1 - x - y)e^y, -x(1 - x - y)e^y, (1 - x - y) \rangle \cdot \langle 1, 1, 1 \rangle$$

Simplifying, we get:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D 1 - x - y \, dA$$

The surface $z = 1 - x - y$ is above the $xy-$ plane in the (now not unexpected) triangle, so

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D 1 - x - y \, dA = \int_0^1 \int_{y=0}^{y=1-x} 1 - x - y \, dy \, dx = \frac{1}{6}$$

Remember to take the negative! So our answer is $-1/6$

16.7, 23 Same kind of thing as before- We might find polar coordinates convenient.

We note that again the desired orientation is downward, but our default orientation is upward, so we will take the negative of our answer.

In this case, the surface is given as the top half of a sphere

$$z = \sqrt{4 - x^2 - y^2} \quad z_x = \frac{-x}{\sqrt{4 - x^2 - y^2}} \quad z_y = \frac{-y}{\sqrt{4 - x^2 - y^2}}$$

over the first quadrant of the xy plane. Therefore, with $\vec{F} = \langle x, -z, y \rangle$, we have:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(\frac{x^2}{\sqrt{4 - x^2 - y^2}} - y + y \right) dA$$

Since D is the quarter circle (radius 2), we will go ahead and do a polar conversion:

$$\int_0^{\pi/2} \int_0^2 \frac{r^2 \cos^2(\theta)}{\sqrt{4 - r^2}} r \, dr \, d\theta = \int_0^{\pi/2} \cos^2(\theta) \, d\theta \cdot \int_0^2 \frac{r^3}{\sqrt{4 - r^2}} \, dr$$

Use the double angle on $\cos^2(\theta)$, and $u = 4 - r^2$ for the second integral:

$$\int_0^{\pi/2} \frac{1}{2} + \frac{1}{2} \cos(2\theta) \, d\theta \cdot \int_4^0 -\frac{1}{2}(4 - u) u^{-1/2} \, du = \cdots = \frac{4\pi}{3}$$

Remember to take the negative of the answer: $-4\pi/3$

16.7, 27 **You do not need to be responsible for this problem.**

The issue: The sides of the cube cannot be represented by functions of the form $z = f(x, y)$. This is easily remedied by the more generic form of parameterization $\vec{r}(u, v)$, but that will not be on the exam.