Quiz 3 Solutions

1. Find the point of intersection (if there is one) between the line given by $x = t$, $y = 3t - 2$, $z = -t$ and the plane given by $x + y + z = 1$.

   The point of intersection is a time at which the point of the line is a point of the plane. Such a point must satisfy the equation of the plane:

   $$(t) + (3t - 2) + (-t) = 1 \quad \Rightarrow \quad 3t = 3 \quad \Rightarrow t = 1$$

   At time 1, the line intersects the plane at the point:

   $$(1, 1, -1)$$

2. Consider the vector $\mathbf{c} = \mathbf{a} \times (\mathbf{a} \times \mathbf{b})$. Is $\mathbf{c} \perp \mathbf{a}$? Is $\mathbf{c} \perp \mathbf{b}$? Explain.

   SOLUTION: Since the vector produced by the cross product is orthogonal to the vectors that created it, $\mathbf{c}$ must be orthogonal to $\mathbf{a}$.

   Is $\mathbf{c}$ necessarily orthogonal to $\mathbf{b}$? Let's try a quick example: Let $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$, just to see what happens:

   $$\mathbf{a} \times \mathbf{b} = \mathbf{k} \qquad \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

   In this case, $\mathbf{c}$ is parallel to $\mathbf{b}$, so the answer is NO.

3. Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors. What is the minimum and maximum magnitude of the cross product, $\mathbf{a} \times \mathbf{b}$?

   $$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$$

   So if $\mathbf{a}, \mathbf{b}$ have unit length, then the magnitude can be any number between 0 and 1 (the values of $\sin(\theta)$, where θ can range between 0 and π).

4. Find the distance between the planes $x + 2y + 1 = z$ and $3x + 6y - 3z = 4$.

   If the planes are not parallel, we are done. Check the normals after putting the planes in standard form:

   $$x + 2y - z + 1 = 0 \qquad 3x + 6y - 3z - 4 = 0$$

   $$\mathbf{n}_1 = < 1, 2, -1 > \qquad \mathbf{n}_2 = < 3, 6, -3 >$$

   They are parallel. To find the distance, we can choose any point on the first plane, and find its distance to the second (or vice-versa): We choose point $(-1, 0, 0)$ on the first plane.

   $$\frac{|-3 + 0 + 0 - 4|}{\sqrt{3^2 + 6^2 + 3^2}} = \frac{7}{\sqrt{54}}$$

MATCHING: 1-VI, 2-I, 3-II, 4-IV, 5-V, 6-III